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Research Article

Fekete-Szegö Like Inequality for Certain Subclasses of Meromorphic Functions with Fixed Residue d

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Abstract: In this paper, the authors introduce certain new subclasses of meromorphic functions with a fixed point ξ defined in the punctured unit disk $\mathbb{U}_{\xi}^{*} := \{z \in \mathbb{C} : 0 < |z - \xi| < 1\}$ and obtain the sharp upper bound for the coefficient functional $|a_1 - \mu a_0^2|$ for the function in this class with respect to fixed residue d. Our result generalizes the result of Aytaş and Güney (2013).

Keywords: Meromorphic function, Subordination, Starlike function, Fekete-Szegö inequality.

1 Introduction and Motivation

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the *open unit* disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$, consisting of functions which are univalent in $\mathbb{U}$. We here recall the definitions of the well-known classes of starlike and convex functions as follows (see [1]):

$$\mathcal{S}^{*} = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\},$$

and

$$\mathcal{S}^c = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\}.$$

Let ξ be a fixed point in $\mathbb{U}$. Let $\mathcal{H}(\mathbb{U})$ be the set of functions which are regular in $\mathbb{U}$ and

$$\mathcal{A}(\xi) = \{f \in \mathcal{H}(\mathbb{U}) : f(\xi) = f'(\xi) - 1 = 0\}.$$

Denote by

$$\mathcal{S}(\xi) = \{f \in \mathcal{A}(\xi) : f \quad \text{is} \quad \text{univalent} \quad \text{in} \quad \mathbb{U}\},$$

the subclass of $\mathcal{A}(\xi)$ consists of the functions of the form:

$$f(z) = (z - \xi) + \sum_{k=2}^{\infty} a_k (z - \xi)^k \tag{1.2}$$

which are analytic and univalent in $\mathbb{U}$. Clearly $\mathcal{S}(0) = \mathcal{S}$.

In 1999, Kanas and Ronning [8] introduced the following classes of functions:

$$\mathcal{ST}(\xi) = \mathcal{S}^*(\xi) = \left\{ f \in \mathcal{S}(\xi) : \Re\left(\frac{(z-\xi)f'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\}, \tag{1.3}$$

$$\mathcal{CV}(\xi) = \mathcal{S}^c(\xi) = \left\{ f \in \mathcal{S}(\xi) : \Re\left(1 + \frac{(z-\xi)f''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\}, \tag{1.4}$$

and these classes were extensively studied by Acu and Owa [1].

The class $\mathcal{S}^*(\xi)$ is defined by the geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{S}^c(\xi)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to ones introduced by Goodman [6, 7] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. It is obvious that there exists a natural "Alexander relation" between the classes $\mathcal{S}^*(\xi)$ and $\mathcal{S}^c(\xi)$:

$$g \in \mathcal{S}^c(\xi) \Longleftrightarrow f(z) = (z - \xi)g'(z) \in \mathcal{S}^*(\xi).$$

Let $\sum$ denote the class of the functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.5}$$

that are regular and univalent on the punctured unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

with a simple pole at origin with residue 1.
For $0 \le \xi < 1$, let $\sum_\xi$ denote the class of functions f which are meromorphic and univalent in the unit disc $\mathbb{U}$ with normalization $\lim_{z \longrightarrow \xi} f(z) = \infty$.

In the punctured open unit disk

$$\mathbb{U}_\xi^* = \{z \in \mathbb{C} : 0 < |z-\xi| < 1\} = \mathbb{U} - \{\xi\},$$

every function $f \in \sum_\xi$ has an expansion of the form

$$f(z) = \frac{d}{z-\xi} + \sum_{k=0}^{\infty} a_k (z-\xi)^k \quad (d \in \mathbb{C} \setminus \{0\}), \tag{1.6}$$

where $d = Res(z, \xi)$.

A function f of the form (1.6) is said to be meromorphic starlike function with fixed residue d, denoted by $\sum_\xi^*$ if

$$-\Re\left(\frac{(z-\xi)f'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{1.7}$$

Similarly, a function $f \in \sum_\xi$ is said to be meromorphic convex function with fixed residue d, denoted by $\sum_\xi^c$ if

$$-\Re\left(1 + \frac{(z-\xi)f''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{1.8}$$

Let $g(z)$ and $f(z)$ be two analytic functions in $\mathbb{U}$. A function $g(z)$ is said to be subordinate to $f(z)$ if there exists an analytic function $\omega(z)$ in the unit disk $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that $g(z) = f(\omega(z))$. We denote this subordination by

$$g(z) \prec f(z) \quad (z \in \mathbb{U}). \tag{1.9}$$

For brief survey on the concept of subordination, see [3].

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{U}$ with $\phi(0) = 1$ and $\phi'(0) > 0$ which maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Such function ϕ has a series expansion of the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \tag{1.10}$$

where $B_1, \quad B_2 \geq 0$ and B_n's are real.

Let $\sum^*(\phi)$ be the class of functions $f \in \sum$ for which

$$-\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathbb{U}).$$

The class $\sum^*(\phi)$ is analogues to the class $\mathcal{S}^*(\phi)$, introduced by Ma and Minda [9].

It is well-known (see [4]) that for $f \in \mathcal{S}$ and given by (1.1), the sharp inequality $|a_3 - a_2^2|$ holds. Fekete-Szegö [5] obtained sharp upper bounds for $|\mu a_2^2 - a_3|$ for $f \in \mathcal{S}$ when μ is real. Thus, the determination of the sharp upper bounds for the nonlinear functional $|a_3 - \mu a_2^2|$ for any compact family $\mathcal{F}$ of functions in $\mathcal{S}$ is popularly known as the Fekete-Szegö problem for $\mathcal{F}$. For brief history of the Fekete-Szegö problem see [12].

Silverman et al. [11] has obtained sharp upper bounds for Fekete-Szegö like functional $|a_1 - \mu a_0^2|$ for certain subclasses of $\sum$. Recently, Aytaş and Güney [2] introduced and studied the class $\sum_\xi^*(\phi)$ of functions $f \in \sum_\xi$ satisfying the condition

$$-\frac{(z-\xi)f'(z)}{f(z)} \prec \phi(z). \tag{1.11}$$

Motivated essentially by the above mentioned works, in this paper the authors introduce certain new subclass of meromorphic function with fixed residue d and obtain the Fekete-Szegö inequality for the function in these class. Our result generalizes the result of Aytaş and Güney (see [2]).

Definition 1.1 *A function $f \in \sum_\xi$ is said to be member of the class $\sum_\xi^{*,\gamma}(\alpha, \beta; \phi)$ if*

$$(1-\beta)\left(\frac{(z-\xi)f(z)}{d}\right)^\alpha + \beta\left(-\frac{(z-\xi)f'(z)}{f(z)}\right)\left(\frac{(z-\xi)f(z)}{d}\right)^\alpha \prec [\phi(z)]^\gamma \tag{1.12}$$

where $0 \leq \alpha < \beta \leq 1, \quad 0 < \gamma \leq 1$ and the powers are taken with their principal values.

It is noted that $\alpha = 0, \beta = 1, \gamma = 1$, the class $\sum_\xi^{*,1}(0, 1, \phi)$ reduces to the class $\sum_\xi^*(\phi)$ studied by Aytaş and Güney (see [2]).

2 Preliminaries

Let $\mathcal{P}(\xi)$ be the class of all functions

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k (z-\xi)^k \tag{2.1}$$

that are regular in $\mathbb{U}$ and satisfy the conditions $p(\xi) = 1$ and $\Re(p(z)) > 0$ for $z \in \mathbb{U}$. We need the following lemmas for our investigation.

Lemma 2.1 *(see [10]) If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{U}$, then for any complex number μ,*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}.$$

The result is sharp for the functions defined by $p(z) = \frac{1+z^2}{1-z^2}$ or $p(z) = \frac{1+z}{1-z}$.

Lemma 2.2 *(see [1, 13]) If $p \in \mathcal{P}(\xi)$,*

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k (z-\xi)^k,$$

then

$$|c_k| \leq \frac{2}{(1+e)(1-e)^k}, \tag{2.2}$$

where $e = |\xi|, \quad k \geq 1$.

3 Main Result

We prove the following result:

Theorem 3.1 *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1, \quad B_2 \geq 0$ and B_n's are real. If $f(z)$ given by (1.6) belongs to $\sum_{\xi}^{*,\gamma}(\alpha, \beta, \phi)$, then for any complex number μ, we have*

$$(i) \quad |a_1 - \mu a_o^2| \leq \frac{\gamma B_1 |d|}{2\beta - \alpha} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\gamma - 1}{2} B_1 - \frac{\gamma B_1 (2\beta - \alpha)}{2(\beta-\alpha)^2}(1 - \alpha - \mu d)\right|\right\} \quad (B_1 \neq 0) \tag{3.1}$$

$$(ii) \quad |a_1 - \mu a_o^2| \leq \frac{2\gamma |d|}{(2\beta - \alpha)(1+e)^2(1-e)^2} \quad (B_1 = 0). \tag{3.2}$$

Proof: Let $f \in \sum_{\xi}^{*,\gamma}(\alpha, \beta; \phi)$. Hence by Definition 1.1 there exists a Schwarz function

$$w(z) = (z-\xi) + A_2(z-\xi)^2 + A_3(z-\xi)^3 + \cdots,$$

analytic in $\mathbb{U}_{\xi}^*$ with $w(\xi) = 0$ and $|w(z)| < 1$ in $\mathbb{U}_{\xi}^*$ such that

$$(1-\beta)\left(\frac{(z-\xi)f(z)}{d}\right)^{\alpha} + \beta\left(\frac{-(z-\xi)f'(z)}{f(z)}\right)\left(\frac{(z-\xi)f(z)}{d}\right)^{\alpha} = [\phi(w(z))]^{\gamma}. \tag{3.3}$$

Define the function $p(z)$ by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1(z-\xi) + c_2(z-\xi)^2 + \cdots \tag{3.4}$$

Since $w(z)$ is a Schwarz's function, we see that $\Re(p(z)) > 0$ and $p(\xi) = 1$. Solving $w(z)$ in terms of $p(z)$ we get

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1(z-\xi) + c_2(z-\xi)^2 + \cdots}{2 + c_1(z-\xi) + c_2(z-\xi)^2 + \cdots}. \tag{3.5}$$

Now

$$\begin{aligned}[\phi(w(z))]^{\gamma} &= \left[1 + \frac{B_1 c_1}{2}(z-\xi) + \left(\frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2 c_1^2\right)(z-\xi)^2 + \cdots\right]^{\gamma} \\ &= 1 + \frac{\gamma B_1 c_1}{2}(z-\xi) + \left[\frac{\gamma B_1}{2}\left(c_2 - \frac{c_1^2}{2} + \frac{\gamma B_2 c_1^2}{4} + \frac{\gamma(\gamma-1)}{2}\frac{B_1^2 c_1^2}{4}\right)\right](z-\xi)^2 + \cdots.\end{aligned} \tag{3.6}$$

On the other hand, a simple computation gives

$$\begin{aligned}(1-\beta)\left(\frac{(z-\xi)f(z)}{d}\right)^{\alpha}+\beta\left(-\frac{(z-\xi)f'(z)}{f(z)}\right)\left(\frac{(z-\xi)f(z)}{d}\right)^{\alpha}\\ =1-(\beta-\alpha)\frac{a_0}{d}(z-\xi)-\frac{2\beta-\alpha}{d}\left(a_1-\frac{1}{2}(1-\alpha)\frac{a_0^2}{d}\right)(z-\xi)^2+\cdots.\end{aligned} \tag{3.7}$$

Using (3.6) and (3.7) in (3.3) we have

$$a_0=-\frac{d}{2(\beta-\alpha)}\gamma B_1c_1, \tag{3.8}$$

and

$$a_1=-\frac{\gamma B_1 d}{2(2\beta-\alpha)}\left[c_2-\frac{c_1^2}{2}\left(1-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1+\frac{B_1\gamma(1-\alpha)(2\beta-\alpha)}{2(\beta-\alpha)^2}\right)\right]. \tag{3.9}$$

For any complex number μ, we have

$$|a_1-\mu a_0^2|=\frac{\gamma B_1|d|}{2(2\beta-\alpha)}\left|c_2-\nu c_1^2\right|, \tag{3.10}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{\gamma-1}{2}B_1+\frac{(2\beta-\alpha)}{2(\beta-\alpha)^2}\gamma B_1(1-\alpha-\mu d)\right]. \tag{3.11}$$

By application of Lemma 2.1 to the right hand side of (3.10), we get the required inequality (3.1).
Case (ii) If $B_1=0$ then from (3.8) and (3.9) we obtain

$$\begin{aligned}a_0&=0\\ a_1&=-\frac{\gamma c_1^2 d}{4(2\beta-\alpha)}B_2.\end{aligned}$$

For any complex number μ, we have

$$|a_1-\mu a_0^2|=\frac{\gamma|d|}{4(2\beta-\alpha)}|B_2c_1^2|. \tag{3.12}$$

Making use of Lemma 2.2 and well-known fact that $|B_2|\le 2$ in (3.12) give

$$|a_1-\mu a_0^2|\le\frac{2\gamma|d|}{(2\beta-\alpha)(1+e)^2(1-e)^2},$$

which proves (ii).
The bounds are sharp i.e.,

$$(1-\beta)\left(\frac{(z-\xi)f(z)}{d}\right)^{\alpha}+\beta\left(\frac{-(z-\xi)f'(z)}{f(z)}\right)\left(\frac{(z-\xi)f(z)}{d}\right)^{\alpha}=[\phi(w(z))]^{\gamma}=\left[\phi(z-\xi)^2\right]^{\gamma},$$

and

$$(1-\beta)\left(\frac{(z-\xi)f(z)}{d}\right)^{\alpha}+\beta\left(\frac{-(z-\xi)f'(z)}{f(z)}\right)\left(\frac{(z-\xi)f(z)}{d}\right)^{\alpha}=[\phi(w(z))]^{\gamma}=[\phi(z-\xi)]^{\gamma}.$$

Remark 3.1 *Taking $\alpha=0$, $\beta=1$ and $\gamma=1$ in Theorem 3.1, we get the result of Aytaş and Güney* (see [2]).

4 Conclusion

In this paper, we have investigated the Fekete-Szegö like inequality problem for the function $f \in \sum_{\xi}$ in the class $\sum_{\xi}^{*,\gamma}(\alpha, \beta; \phi)$ defined as (1.12). One can define the class as

$$\frac{-\left(\frac{z-\xi}{d}\right) f'(z)}{(1-\lambda)\frac{f(z)}{d} + \lambda\left(-\frac{(z-\xi)f'(z)}{d}\right)} \prec [\phi(z)]^{\gamma} \quad (0 \leq \lambda < 1, \quad 0 < \gamma \leq 1).$$

Proceeding as Theorem 3.1, it left as an open problem for researcher to find out the Fekete-Szegö inequality in the context of the modified class. Putting $\lambda = 0$ and $\gamma = 1$ in the modified result, we get the result due to Aytaş and Güney (see [2]).

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