Information in **S**ciences and **C**omputing
Volume 2015, Article ID ISC530515, 15 pages
Available online at http://www.infoscicomp.com/

Research Article
Localization Principle in Normed Quasilinear Spaces

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Received 29 May 2015; Accepted 24 July 2015

Abstract: Aseev [1] has launched a new branch of functional analysis by introducing the concepts of quasilinear spaces and normed quasilinear spaces which is generalization of classical linear spaces and normed linear spaces. After being introduced of normed quasilinear spaces, it is natural and even necessary to research topological properties of normed quasilinear spaces. This study serves to this purpose by giving a general method to construct a topology on any quasilinear space. To achieve this, we present some definitions in quasilinear spaces such as balanced set, absorbing set and additivity. Also we obtain some results related to these concepts. Moreover, in this paper, we observe that the elegance property of normed linear spaces namely "localization principle" may not holds in normed quasilinear spaces. One of the main purposes of this paper is to give a result about localization principle in normed quasilinear spaces. This is especially important for further progression of some matters about topology of normed quasilinear spaces. Our researches draw a border to this problem with a new concept entitled "stuff of elements" in quasilinear spaces.

Keywords: Quasilinear spaces; Normed quasilinear spaces; Localization principle; Stuff; Quasilinear localization.

1 Introduction

In 1986, Aseev [1] introduced the concept of quasilinear spaces including both classical linear spaces and nonlinear spaces. Then he proceeded a similar way to linear functional analysis on quasilinear spaces by introducing the notions of the norm, quasilinear operator and functional. Further, he presented some results which are quasilinear counterparts of fundamental definitions and theorems in linear functional analysis and differential and integral calculus in Banach spaces.

This vanguard work brings some different aspects to set-valued algebra and analysis by the advantages of the partial order relation and encourages us to introduce some new results in this field, [2, 3, 4, 5, 6].

Recall that, the localization principle states that: in a normed linear space X, if $U \subset X$ and $\theta \in U$ then U is a neighbourhood of θ if and only if $x + U$ is a neighbourhood of x.

One of the main purposes of this study is to state and prove localization principle in normed quasilinear spaces.

2 Preliminaries and Some Results on Quasilinear Spaces and Normed Quasilinear Spaces

Definition 2.1 *[1] A set X is called quasilinear space (qls, for short), if a partial order relation "$\preceq$", an algebraic sum operation and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for all elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$:*

$$x \preceq x, \tag{1}$$

$$x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \tag{2}$$

$$x = y \text{ if } x \preceq y \text{ and } y \preceq x, \tag{3}$$

$$x + y = y + x, \tag{4}$$

$$x + (y + z) = (x + y) + z, \tag{5}$$

$$\text{there exists an element } \theta \in X \text{ such that } x + \theta = x, \tag{6}$$

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta)x, \tag{7}$$

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \tag{8}$$

$$1 \cdot x = x, \tag{9}$$

$$0 \cdot x = \theta, \tag{10}$$

$$(\alpha + \beta) \cdot x \preceq \alpha \cdot x + \beta \cdot x, \tag{11}$$

$$x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \tag{12}$$

$$\alpha \cdot x \preceq \alpha \cdot y \text{ if } x \preceq y. \tag{13}$$

The qls just defined is denoted by $(X, \preceq)$ or simply by X.

A linear space is a qls with the partial order relation "$=$".

The most popular example of qls which is not a linear space is the set of all nonempty, compact and convex subsets of real numbers with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and multiplication operation by a real number λ defined by

$$\lambda \cdot A = \{\lambda \cdot a : a \in A\}.$$

We denote this set by $\Omega_C(\mathbb{R})$.

Another one is $\Omega(\mathbb{R})$ which is the set of all nonempty and compact subsets of real numbers.

In general, $\Omega(E)$ stands for family of all nonempty, closed and bounded subsets of any normed linear space E and $\Omega_C(E)$ denotes family of nonempty, convex, closed and bounded subsets of E. Although $\Omega(E)$ and $\Omega_C(E)$ are not linear space, both of them are quasilinear spaces with the inclusion relation and a slight modification of addition operation by

$$A+B=\overline{\{a+b : a\in A,\ b\in B\}}$$

and multiplication by $\lambda\in\mathbb{R}$ defined by $\lambda\cdot A=\{\lambda\cdot a : a\in A\}$.

Lemma 2.2 *[1] In a qls $(X,\preceq)$, the element θ is minimal, i.e., $x=\theta$ if $x\preceq\theta$.*

We note that the minimality is not only a property of θ but also may be shared by some other elements.

An element x' is called inverse of $x\in X$ if $x+x'=\theta$. Further, if inverse element exists then it is unique. An element x possessing inverse is called regular, otherwise is called singular. X_r and X_s denote the sets of all regular and singular elements in X, respectively.

It will be assumed throughout the paper that $-x=(-1)\cdot x$.

Proposition 2.3 *[6] In a qls $(X,\preceq)$, every regular element is minimal.*

Suppose that every element x has inverse element $x'\in X$. Then the partial order in X is determined by equality, the distributivity condition in (11) holds with the equality and consequently, X is a linear space, [1]. In a real linear space, "=" is only way to define a partial order such that the conditions (1)-(13) hold.

Let X be a qls and $Y\subseteq X$. Then Y is called a subspace of X whenever Y is a qls with the same partial order and the restriction to Y of the operations on X.

Theorem 2.4 *[6] Let $(X,\preceq)$ be a qls and $Y\subseteq X$. Then Y is a subspace of X if and only if $\alpha\cdot x+\beta\cdot y\in Y$ for every $x,y\in Y$ and $\alpha,\beta\in\mathbb{R}$.*

An element $x\in X$ is said to be symmetric if $-x=x$. X_d denotes the set of all symmetric elements of X.

Also X_r, X_d and $X_s\cup\{0\}$ are subspaces of X and these subspaces are called regular, symmetric and singular subspaces of X, respectively.

For example, let $X=\Omega_C(\mathbb{R})$ and $Z=\{0\}\cup\{[a,b] : a,b\in\mathbb{R} \text{ and } a<b\}$. Then Z is singular subspace of X. On the other hand, the set of all singletons of real numbers $\{\{a\} : a\in\mathbb{R}\}$ is regular subspace of X, [6].

Definition 2.5 *[1] Let $(X,\preceq)$ be a qls. A real function $\|\cdot\|_X : X\longrightarrow\mathbb{R}$ is called a norm if the following conditions hold:*

$$\|x\|_X>0 \text{ if } x\neq\theta, \tag{14}$$

$$\|x+y\|_X \le \|x\|_X + \|y\|_X\,, \tag{15}$$
$$\|\alpha \cdot x\|_X = |\alpha|\,\|x\|_X\,, \tag{16}$$
$$\|x\|_X \le \|y\|_X\,,\ \textit{if } x \preceq y, \tag{17}$$

$$\begin{gathered}\textit{if for any } \varepsilon > 0 \textit{ there exists an element } x_\varepsilon \in X \textit{ such that}\\ x \preceq y + x_\varepsilon \textit{ and } \|x_\varepsilon\|_X \le \varepsilon \textit{ then } x \preceq y.\end{gathered} \tag{18}$$

A normed quasilinear space (briefly, normed qls) X is a qls with a norm defined on it.

If every element x has inverse element $x' \in X$, then the concept of normed qls coincides with the notion of real normed linear space, [1].

Let $(X, \preceq)$ be a normed qls. Hausdorff metric on X is defined by the equality

$$h_X(x,y) = \inf\left\{r \ge 0 : x \preceq y + a_1^{(r)},\ y \preceq x + a_2^{(r)} \text{ and } \left\|a_i^{(r)}\right\| \le r,\ i = 1,2\right\}.$$

Since $x \preceq y + (x-y)$ and $y \preceq x + (y-x)$, the quantity $h_X(x,y)$ is well defined for any elements $x, y \in X$. Also, it is not hard to see that the function $h_X(x,y)$ satisfies all metric axioms. We should note that $h_X(x,y)$ may not equal to $\|x-y\|_X$ if X is not a linear space; but the inequality

$$h_X(x,y) \le \|x-y\|_X$$

is satisfied for any elements $x, y \in X$. Therefore, we use the metric instead of the norm to discuss topological properties in normed quasilinear spaces. For example, let (x_n) be a sequence in X and $x \in X$. Then $x_n \to x$ if and only if $h_X(x_n, x) \to 0$. However $x_n \to x$ may not implies $\|x_n - x\|_X \to 0$. On the other hand, always $\|x_n - x\|_X \to 0$ implies $x_n \to x$ in normed quasilinear spaces.

Let X be a real Banach space. Then X is a complete normed qls with partial order relation given by equality. Conversely, if X is a complete normed qls and every $x \in X$ has inverse element $x' \in X$ then X is a real Banach space. Also the partial order on X is equality. In this case $h_X(x,y) = \|x-y\|_X$, [1].

Let E be a real normed linear space. Then $\Omega(E)$ and $\Omega_C(E)$ are normed quasilinear spaces with the norm is defined by

$$\|A\|_\Omega = \sup_{a \in A} \|a\|_E\,. \tag{19}$$

In this case, the Hausdorff metric is defined as usual:

$$h_\Omega(A,B) = \inf\{r \ge 0 : A \subseteq B + S(\theta,r),\ B \subseteq A + S(\theta,r)\},$$

where $S(\theta, r)$ is the closed ball of radius r and centered at $\theta \in E$, [1].

Lemma 2.6 *[1] The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is a continuous function with respect to the Hausdorff metric.*

In $\Omega_C(\mathbb{R})$, the distance between two intervals $x = [x_1, x_2]$ and $y = [y_1, y_2]$ can be easily calculated with the following metric, [7]

$$h(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

3 Some New Results About Quasilinear Spaces

Definition 3.1 *Let $(X, \preceq)$ be a qls. A subset $U \subset X$ is called balanced if $t \cdot U \subset U$ for every $|t| \leq 1$.*

Example 3.2 *The unit ball $S(\{0\}, 1)$ of $\Omega_C(\mathbb{R})$ is a balanced set.*

To show this, let us take an arbitrary element y in $t \cdot S(\{0\}, 1)$. Then there exists an element $z = [z_1, z_2] \in S(\{0\}, 1)$ such that $y = t \cdot z$. Thus we say

$$h(z, \{0\}) = \max\{|z_1 - 0|, |z_2 - 0|\} \leq 1.$$

This implies $z \subseteq [-1, 1]$. Since $|t| \leq 1$ we have

$$y = t \cdot z \subseteq t \cdot [-1, 1] = [-t, t] \subseteq [-1, 1]$$

for every $z \in S(\{0\}, 1)$. Hence $y \in S(\{0\}, 1)$.

The set $A = \{x \in \Omega_C(\mathbb{R}) : x \subseteq [-2, 3]\}$ in $\Omega_C(\mathbb{R})$ is not balanced since $t \cdot x = [-3, 2] \notin A$ for $t = -1$ and $x = [-2, 3] \in A$.

Further, the sets $S\left(\{2\}, \frac{1}{2}\right)$ and $S([-2, 3], 4)$ in $\Omega_C(\mathbb{R})$ are not balanced.

Also, let $B = \{x \in (\Omega_C(\mathbb{R}))_r : x \subseteq [-2, 1]\}$ and $U = \{x : x \subseteq [-2, 1]$ and $x \notin B\}$. Then U is an example of balanced set which is not a sphere.

Example 3.3 *The ball $S([-1, 1], 1)$ is balanced although $S\left([-1, 1], \frac{1}{4}\right)$ and $S([-1, 1], 0.99)$ are not balanced in $\Omega_C(\mathbb{R})$.*

Since

$$h\left(\left[-\frac{1}{2}, \frac{1}{2}\right], [-1, 1]\right) = \max\left\{\left|-\frac{1}{2} + 1\right|, \left|\frac{1}{2} - 1\right|\right\} = \frac{1}{2} \not\leq \frac{1}{4},$$

we obtain

$$t \cdot x = \left[-\frac{1}{2}, \frac{1}{2}\right] \notin S\left([-1, 1], \frac{1}{4}\right)$$

for $t = \frac{1}{2}$ and $x = [-1, 1] \in S\left([-1, 1], \frac{1}{4}\right)$. So the ball $S\left([-1, 1], \frac{1}{4}\right)$ is not balanced.

If it is considered with the same center but radius $r = 0.99$, the ball $S([-1, 1], 0.99)$ is not balanced. Indeed we write

$$t \cdot x = [0, 0] \notin S([-1, 1], 0.99)$$

for every $x \in S([-1, 1], 0.99)$ and $t = 0$.

Now we consider the same centered ball with radius $r = 1$. Then it becomes a balanced set.

Let $[a, b] \in S([-1, 1], 1)$. Then we write

$$h([a, b], [-1, 1]) = \max\{|a + 1|, |b - 1|\} \leq 1.$$

So we say

$$|a + 1| \leq 1 \text{ and } |b - 1| \leq 1$$

that is

$$-2 \leq a \leq 0 \text{ and } 0 \leq b \leq 2.$$

We want to show that the including $t \cdot S([-1,1],1) \subseteq S([-1,1],1)$ *holds for every* t *such that* $|t| \leq 1$.

♦ *For every* $t \in [-1,0]$, *we write* $t \cdot [a,b] = [tb, ta]$.

Also, since

$$-2 \leq a \leq 0 \text{ and } 0 \leq b \leq 2$$

we have

$$-1 \leq ta - 1 \leq 1 \text{ and } -1 \leq tb + 1 \leq 1.$$

These imply that respectively

$$|ta - 1| \leq 1 \text{ and } |tb + 1| \leq 1.$$

Thus we can say

$$h([tb, ta], [-1,1]) = \max\{|tb+1|, |ta-1|\} \leq 1$$

and

$$t \cdot [a,b] = [tb, ta] \in S([-1,1],1).$$

♦ *With the similar arguments we can prove that*

$$h([ta, tb], [-1,1]) = \max\{|ta+1|, |tb-1|\} \leq 1$$

for every $t \in [0,1]$. *So we obtain*

$$t \cdot [a,b] = [ta, tb] \in S([-1,1],1).$$

Hence the including $t \cdot S([-1,1],1) \subseteq S([-1,1],1)$ *holds for every* $|t| \leq 1$. *Therefore the ball* $S([-1,1],1)$ *is balanced.*

With similar way, one can see that every the ball $S([-1,1],r)$ *such that* $r \geq 1$ *is balanced.*

This example illustrates that whether the balls centered at a symmetric element is balanced or not depends on the radius of the ball.

According to this, we can give the following result:

Corollary 3.4 *For the ball* $S([-a,a],r)$ *in* $\Omega_C(\mathbb{R})$*;*

$$\text{the ball is not balanced, if } r < a,$$
$$\text{the ball is balanced, if } r \geq a.$$

Proof Firstly let us prove that $S([-a,a],r)$ is not balanced when $r < a$:
Let $[x,y] \in S([-a,a],r)$. For $t = 0$, we have

$$t \cdot [x,y] = [0,0] \notin S([-a,a],r)$$

since

$$h([0,0],[-a,a]) = a > r.$$

Now, let us prove that $S([-a,a],r)$ is balanced when $r \geq a$:

For any element $[x,y] \in S([-a,a],r)$, we write

$$h([x,y],[-a,a]) = \max\{|x+a|,|y-a|\} \leq r.$$

This implies

$$|x+a| \leq r \text{ and } |y-a| \leq r$$

that is

$$-r-a \leq x \leq r-a \text{ and } -r+a \leq y \leq r+a.$$

We want to show that the including $t \cdot S([-a,a],r) \subseteq S([-a,a],r)$ holds for every $|t| \leq 1$.

♦ For every $t \in [-1,0]$, we write $t \cdot [x,y] = [ty,tx]$.

Also, since $-r-a \leq x \leq r-a$ and $-r+a \leq y \leq r+a$ we obtain

$$-r \leq ty + a \leq r \text{ and } -r \leq tx - a \leq r$$

that is

$$|ty+a| \leq r \text{ and } |tx-a| \leq r.$$

Hence we obtain

$$h([ty,tx],[-a,a]) = \max\{|ty+a|,|tx-a|\} \leq r$$

and

$$t \cdot [x,y] = [ty,tx] \in S([-a,a],r).$$

♦ With the similar discussions, we can prove that

$$h([tx,ty],[-a,a]) = \max\{|tx+a|,|ty-a|\} \leq r$$

for every $t \in [0,1]$. Thus it can be written

$$t \cdot [x,y] = [tx,ty] \in S([-a,a],r).$$

So, if $r \geq a$, the including $t \cdot S([-a,a],r) \subseteq S([-a,a],r)$ holds for every $|t| \leq 1$.

Consequently, the ball $S([-a,a],r)$ is balanced when $r \geq a$. ■

Definition 3.5 *Let $(X,\preceq)$ be a qls. A subset $U \subset X$ is called absorbing if for every $x \in X$ there exists $\varepsilon > 0$ such that $t \cdot x \in U$ for $|t| \leq \varepsilon$.*

Example 3.6 *The unit ball $S(\{0\},1)$ of $\Omega_C(\mathbb{R})$ is an absorbing set.*

Also, the set

$$U = S(\{0\},\frac{1}{4}) \cup \{[-2,2],\ [-3,3]\}$$

is an example of absorbing set which is not balanced. Indeed, this set is not balanced since

$$t \cdot x = \left[-\frac{1}{2}, \frac{1}{2} \right] \notin U$$

for $t = \frac{1}{4}$ *and* $x = [-2, 2] \in U$.

Now we show that the set U *is absorbing. Let* x *be an arbitrary element in* $\Omega_C(\mathbb{R})$. *If we choose* $\varepsilon = 1/5 \|x\|$, *then* $t \cdot x \in S(\{0\}, \frac{1}{4})$ *for every* $|t| \leq \varepsilon$. *Really,*

$$h(t \cdot x, \{0\}) = \|t \cdot x\| = |t| \, \|x\| \leq \frac{1}{5} < \frac{1}{4}.$$

Thus $t \cdot x \in U$ *and* U *is an absorbing set.*

Definition 3.7 *A collection* $\mathcal{F}$ *of subsets of a quasilinear space is called additive if for every* $U \in \mathcal{F}$ *there exists* $V \in \mathcal{F}$ *with* $V + V \subset U$.

Example 3.8 *Let us consider the families* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *in* $\Omega_C(\mathbb{R})$ *defined as* $\mathcal{F}_1 = \{\{x\} : x \in \mathbb{R}\}$ *and* $\mathcal{F}_2 = \{[a, a+1] : a \in \mathbb{R}\}$, *respectively. Then* $\mathcal{F}_1$ *is additive but* $\mathcal{F}_2$ *is not additive.*

Further, let us define

$$U_{x,y} = \{[a, b] : \{x\} \subseteq [a, b] \subseteq [x, x+y]\}.$$

Then the collection $\mathcal{F} = \{U_{x,y} : x, y \in \mathbb{R}\}$ *is additive.*

Definition 3.9 *[8] Let* X *be a set. A filter on* X *is nonempty collection* $\mathcal{F}$ *of nonempty subsets of* X *such that* $A \in \mathcal{F}$, $B \in \mathcal{F}$ *implies* $A \cap B \in \mathcal{F}$, *and* $B \supset A \in \mathcal{F}$ *implies* $B \in \mathcal{F}$.

A filterbase on X *is a nonempty collection* $\mathcal{B}$ *of nonempty subsets of* X *such that* $\mathcal{B}$ *is directed by inclusion.*

Note that we are inspired from the Theorem 4-3-5 in [8] in the following theorem showed a general method of construction a topology on any qls.

Theorem 3.10 *Let* X *be a normed qls and* $\mathcal{B}$ *be an additive filterbase of balanced absorbing subsets of* X. *Then there is a unique topology on* X *for which* $\mathcal{B}$ *is a local base of neighbourhoods of* θ.

Proof Let us define the family τ as

$$\tau = \{G \subset X : x + U \subset G \text{ for every } x \in G \text{ and some } U \subset \mathcal{B}\}.$$

The intersection of two any open sets G_1, G_2 is open:

Let $x \in G = G_1 \cap G_2$. Then there exists $U_1, U_2 \in \mathcal{B}$ such that $x + U_1 \subset G_1$ and $x + U_2 \subset G_2$, respectively. Let $U \in \mathcal{B}$ with $U \subset U_1 \cap U_2$. Then

$$x + U \subset G$$

and so $G = G_1 \cap G_2 \in \tau$.

Let I is an arbitrary index set, $G_i \in \tau$ for every $i \in I$ and $G = \bigcup_{i \in I} G_i$. Then for any $x \in G$ there exists a subset J of I such that $x \in G_i$ for $i \in J$. So there are some $U_i \in \mathcal{B}$ such that $x + U_i \subset G_i$ for $i \in J$. If we take $U = \bigcup_{i \in J} U_i$ then we obtain $x + U \subset \bigcup_{i \in J} G_i \subset G$. This means that $G \in \tau$.

Also $\emptyset, X \in \tau$. Consequently, the family τ defines a topology on X.

Now we show that $\mathcal{B}$ is a local base of neighbourhoods of θ.

Certainly, every neighbourhood H of θ includes an open set G containing θ. Also, since G is open, there exist some $U \in \mathcal{B}$ such that

$$\theta + U = U \subset G \subset H.$$

On the other hand, we must not forget to prove that $\mathcal{B} \subset \mathcal{N}$, that is, the members of $\mathcal{B}$ are neighbourhoods of θ :

Let $U \in \mathcal{B}$ and let us define A by

$$A = \{x : x + V \subset U \text{ for some } V \in \mathcal{B}\}.$$

We prove that $\theta \in A$, $A \subset U$ and A is open.

(i) We obtain $\theta \in A$, since $\theta + U \subset U$;

(ii) Taking into account $\theta \in V$, since V is absorbing, $x \in A$ implies

$$x = x + \theta \in x + V \subset U$$

for some $V \in \mathcal{B}$. So we get $A \subset U$.

(iii) Finally, it remains to prove that A is open. Let $x \in A$. Then we have $x + V \subset U$ for some $V \in \mathcal{B}$ from definition of A. Let $W \in \mathcal{B}$ such that $W + W \subset V$. When we prove that $x + W \subset A$, we will say that A is open. For every $t \in x + W$ there exists an element $w \in W$ such that $t = x + w$. On the other hand, since W is an absorbing set, we can write $\theta \in W$ and

$$t = t + \theta = x + w + \theta \in \underbrace{x + w}_{=t} + W \subset x + W + W \subset x + V \subset U.$$

Hence $t \in A$.

Lastly, the topology is unique since $\mathcal{B}$ generates a unique filter and this is $\mathcal{N}$. This step completes the proof. ■

3.1 Localization Principle in Normed Quasilinear Spaces

We start this section by giving the following definition.

Definition 3.11 *Let X be a normed qls, x be an element of X and h_X be the metric induced by the norm on X. Then, the nonnegative quantity $h_X(x - x, \theta)$ is called diameter of x and denoted by $diam(x)$.*

For each regular element x, $diam\,(x) = 0$ since $x - x = \theta$. Hence this definition is redundant in normed linear spaces since the diameter of every element is 0.

Also it should be emphasized that this notion should not be confused with the classical definition of diameter of a set defined by

$$\delta\,(U) = \sup_{x,y\in U} h_X\,(x,y)$$

for any subset U of the metric space X. For example, we consider the element $[-1,3]$ in $\Omega_C(\mathbb{R})$.

$$\begin{aligned} diam\,([-1,3]) &= h\,([-1,3]-[-1,3]\,,\theta) \\ &= h\,([-4,4]\,,\theta) \\ &= \|[-4,4]\| \\ &= \sup_{a\in[-4,4]} |a| \\ &= 4. \end{aligned}$$

However, for the subset (singleton) $U = \{[-1,3]\}$ of $\Omega_C(\mathbb{R})$

$$\delta\,(U) = \sup_{x,y\in U} h\,(x,y) = h\,([-1,3]\,,[-1,3]) = 0.$$

After this, the family of all neighbourhoods of x will be denoted by $\mathcal{N}_x$.

In this part, we will show that normed quasilinear spaces may not satisfy the localization principle which is a very important property provided by the normed linear space. The following theorem is the provided half of the localization principle in normed quasilinear spaces.

Theorem 3.12 *Let X be a normed qls, $x \in X$ and U be a set containing θ. If $x + U \in \mathcal{N}_x$ then $U \in \mathcal{N}_\theta$.*

Since proof of this theorem is similar to its classical, we will not deal with it.

Although the converse of this theorem is true in normed linear spaces, it may not be true in normed quasilinear spaces. The following example shows this situation.

Example 3.13 *Consider the closed unit ball $S(\{0\}\,,1)$ of $\Omega_C(\mathbb{R})$. Then the set $S(\{0\}\,,1)$ is a neighbourhood of $\{0\}$, but $[2,3] + S(\{0\}\,,1)$ is not a neighbourhood of $[2,3]$:*

We assume that the set $[2,3] + S(\{0\}\,,1)$ is a neighbourhood of $[2,3]$. Then there exists $r > 0$ such that

$$S([2,3],r) \subset [2,3] + S(\{0\}\,,1).$$

On the other hand, diameter of $[2,3]$ is

$$\begin{aligned} \rho\,([2,3]) &= h\,([2,3]-[2,3],\{0\}) \\ &= h\,([-1,1]\,,\{0\}) \\ &= 1. \end{aligned}$$

Now we show that $[2,3]+S(\{0\},1)$ *doesn't contain elements of which diameter is smaller than diameter of* $[2,3]$. *To do this, we take any element* $[a,b]\in S(\{0\},1)$. *Then we write*

$$h([a,b],\{0\})=\max\{|a|,|b|\}\leq 1.$$

This implies

$$-1\leq a\leq 1 \text{ and } -1\leq b\leq 1.$$

Let $[a,b]$ *be any fixed element of* $S(\{0\},1)$. *Then* $[2+a,3+b]\in[2,3]+S(\{0\},1)$. *Also*

$$\begin{aligned}\rho([2+a,3+b]) &= h([2+a,3+b]-[2+a,3+b],\{0\})\\ &= h([a-b-1,b-a+1],\{0\})\\ &= \max\{|a-b-1|,|b-a+1|\}.\end{aligned}$$

Since $a\leq b$, *we write*

$$a-b-1\leq -1 \text{ and } 1\leq b-a+1. \tag{20}$$

On the other hand, the inequalities

$$-1\leq a\leq 1 \text{ and } -1\leq b\leq 1$$

imply

$$-3\leq a-b-1\leq 1 \text{ and } -1\leq b-a+1\leq 3. \tag{21}$$

From (20) and (21), we get

$$-3\leq a-b-1\leq -1 \text{ and } 1\leq b-a+1\leq 3.$$

Therefore we obtain

$$\rho([2+a,3+b])\geq 1.$$

Hence $[2,3]+S(\{0\},1)$ *doesn't contain elements (intervals) of which diameter is smaller than diameter of* $[2,3]$.

However, every ball $S([2,3],r)$ *contains some singletons if* $r\geq diam([2,3])/2=1/2$. *Also, if* $r<diam([2,3])/2=1/2$ *then every ball* $S([2,3],r)$ *contains some singletons and intervals* $[2+r',3-r']$ *such that* $0\leq r'\leq r$.

Further, for $0\leq r'\leq r$, *diameters of the elements* $[2+r',3-r']$ *are smaller than diameter of* $[2,3]$. *Indeed, since* $2+r'\leq 3-r'$ *for the interval* $[2+r',3-r']$, *we write* $r'\leq\frac{1}{2}$. *Then* $0\leq r'\leq\frac{1}{2}$ *implies* $|1-2r'|\leq 1$. *So we obtain*

$$\begin{aligned} & h\left(\left[2+r',3-r'\right]-\left[2+r',3-r'\right],\{0\}\right)\\ = \ & h\left(\left[2r'-1,1-2r'\right],\{0\}\right)\\ = \ & \left|1-2r'\right|\leq 1.\end{aligned}$$

Taking into account that diameters of singletons and elements $[2+r',3-r']$ *such that* $0\leq r'\leq r$ *are smaller than diameter of* $[2,3]$, *it can be concluded that*

$$S([2,3],r)\nsubseteq[2,3]+S(\{0\},1)$$

for every $r>0$. *This result means that the set* $[2,3]+S(\{0\},1)$ *is not a neighbourhood of* $[2,3]$.

So, the localization principle may not be satisfied when a set which is a neighbourhood of $\{0\}$ is translated with any singular element of the normed qls $\Omega_C(\mathbb{R})$. Moreover, this case may be also occur in any normed qls X.

In normed quasilinear spaces, the following result can be given.

Theorem 3.14 *Let X be a normed qls and $x \in X_r$. Then $U \in \mathcal{N}_\theta \Leftrightarrow x+U \in \mathcal{N}_x$.*

Proof Let $f_x : X \to X$ be a translation operator defined by $f_x(v) = v + x$ for a fixed element $x \in X_r$. Then f_x is continuous by the continuity of the algebraic sum operation. Also, inverse of f_x $f_x^{-1}(v) = v - x$ is continuous. Hence the map f_x is a homeomorphism from X onto itself and so it preserves neighbourhoods. ■

Our following viewpoint draws a border to problem "How can the localization principle be expressed in normed quasilinear spaces?" with a new concept: "stuff of elements".

Definition 3.15 (Stuff) *Let $(X, \preceq)$ be a qls and $x \in X$. The set*

$$\{y \in X : y \preceq x\}$$

is called stuff of x and denoted by $\overleftarrow{x}$.

For any $A \subseteq X$, stuff of A is denoted by $\overleftarrow{A}$ and

$$\overleftarrow{A} = \bigcup_{x \in A} \overleftarrow{x}.$$

By the minimality of regular elements, $\overleftarrow{x} = \{x\}$ for any $x \in X_r$ and so for any $A \subset X_r$, $\overleftarrow{A} = A$.

Example 3.16 *In Example 3.13, we have proved that $[2,3] + S(\{0\}, 1)$ is not a neighbourhood of $[2,3]$. Now, we prove that the set $\overleftarrow{[2,3] + S(\{0\}, 1)}$ is a neighbourhood of $[2,3]$. The set*

$$\overleftarrow{[2,3] + S(\{0\}, 1)} = \bigcup_{z \in [2,3]+S(\{0\},1)} \overleftarrow{z}$$

includes the ball $S([2,3], 1)$, that is, $S([2,3], 1) \subset \overleftarrow{[2,3] + S(\{0\}, 1)}$. Indeed; if we take $s = [s_1, s_2] \in S([2,3], 1)$ then

$$h([s_1, s_2], [2,3]) = \max\{|s_1 - 2|, |s_2 - 3|\} \leq 1.$$

This implies that

$$|s_1 - 2| \leq 1 \text{ and } |s_2 - 3| \leq 1$$

that is

$$1 \leq s_1 \leq 3 \text{ and } 2 \leq s_2 \leq 4.$$

We want to show that $s = [s_1, s_2] \in \overleftarrow{[2,3] + S(\{0\},1)}$. *To do this, we must to show that* $s \preceq z$ *for an element* $z \in [2,3] + S(\{0\},1)$.

Let us consider the element $z = [1,4]$. *Then* $z \in [2,3] + S(\{0\},1)$, *since* $z = [1,4] = [2,3] + [-1,1]$ *for the element* $[-1,1]$ *of* $S(\{0\},1)$.

On the other hand, since $1 \leq s_1 \leq s_2 \leq 4$, $[s_1, s_2] \subseteq [1,4]$ *that is* $s \preceq z$. *Consequently, we obtain* $s \in \overleftarrow{[2,3] + S(\{0\},1)}$. *So, the set* $\overleftarrow{[2,3] + S(\{0\},1)}$ *is a neighbourhood of* $[2,3]$.

More generally, one side of localization principle not provided in a normed qls can be presented with the following improvement:

Theorem 3.17 *Let* $(X, \preceq)$ *be a normed qls,* $x \in X$ *and* $D(\theta, r_1)$ *denote open or closed ball of* X. *Then*

$$\overleftarrow{x + D(\theta, r_1)} \in \mathcal{N}_x.$$

Proof Now we show that $B(x, r_1) \subset \overleftarrow{x + D(\theta, r_1)}$, where $B(x, r_1)$ is open ball of X.

If we take $z \in B(x, r_1)$ then

$$\begin{aligned} h(z,x) &= \inf\left\{r \geq 0 : z \preceq x + a_1^{(r)},\ x \preceq z + a_2^{(r)} \text{ and } \left\|a_i^{(r)}\right\| \leq r,\ i = 1,2\right\} \\ &< r_1. \end{aligned} \tag{22}$$

Sake for brevity, let us write

$$\left\{r \geq 0 : z \preceq x + a_1^{(r)},\ x \preceq z + a_2^{(r)} \text{ and } \left\|a_i^{(r)}\right\| \leq r,\ i = 1,2\right\} = A.$$

By the definition of infimum, for every $\varepsilon > 0$ there exists a real number $r \in A$ such that

$$r < \inf A + \varepsilon. \tag{23}$$

Let us take a fixed ε such that

$$0 < \varepsilon < r_1 - \inf A.$$

For this ε, (23) holds. Then it can be found $r \in A$ such that

$$r < \inf A + \varepsilon < \inf A + r_1 - \inf A = r_1.$$

Since $h(z,x) = \inf A$, by using definition of infimum, we obtain $\inf A \leq r$ for every $r \in A$. Hence, there exists a real number r such that

$$h(z,x) \leq r < r_1.$$

According to this, there exists an element $a_1^{(r)} \in X$ such that

$$z \preceq x + a_1^{(r)} \text{ and } \left\|a_1^{(r)}\right\| \leq r < r_1$$

from definition of Hausdorff metric. Thus, we write $z \in \overleftarrow{x + D(\theta, r_1)}$ since $a_1^{(r)} \in D(\theta, r_1)$. So $B(x, r_1) \subset \overleftarrow{x + D(\theta, r_1)}$, that is $\overleftarrow{x + D(\theta, r_1)} \in \mathcal{N}_x$. ■

The following theorem is generalized form of Theorem 3.17.

Theorem 3.18 (Quasilinear Localization) *Let X be a normed qls, $\mathcal{B}$ be an additive filterbase of balanced and absorbing subsets of X, $a \in X$ and $U \in \mathcal{B}$. Then*

$$U \in \mathcal{B} \textit{ implies } \overleftarrow{a+U} \in \mathcal{N}_a. \tag{24}$$

Proof Firstly we recall that the family

$$\tau = \{G \subset X : x + U \subset G \text{ for every } x \in G \text{ and some } U \subset \mathcal{B}\}$$

is a topology on X.

To show that $\overleftarrow{a+U}$ is a neighbourhood of a, we must determine an open G_a containing element a such that $G_a \subset \overleftarrow{a+U}$.

Let us define G_a by

$$G_a = \{x : x + V \subset a + U \text{ for some } V \in \mathcal{B}\}.$$

Now we should prove that $a \in G_a$, $G_a \subset \overleftarrow{a+U}$ and G_a is open.

(i) $a \in G_a$ since $a + U \subset a + U$ and $U \in \mathcal{B}$.

(ii) If we take $z \in G_a$ then there exists a set $V \in \mathcal{B}$ such that $z + V \subset a + U$. Also V contains θ since $V \in \mathcal{B}$ is an absorbing set by the hypotesis. Hence $z \in a+U$ for every element $z \in G_a$. Indeed,

$$z = z + \theta \in z + V \subset a + U.$$

So, we obtain $z \in \overleftarrow{a+U}$ for every $z \in G_a$, since $a + U \subseteq \overleftarrow{a+U}$. Thus $G_a \subset \overleftarrow{a+U}$.

(iii) Let $z \in G_a$. From the definition of G_a, we have $z + V \subset a + U$ for some $V \in \mathcal{B}$. Let $W \in \mathcal{B}$ such that $W + W \subset V$. So, we say that G_a is open when we prove that $z + W \subset G_a$. To do this, if we take $t \in z + W$ then

$$t + W \subset z + W + W \subseteq z + V \subset a + U$$

and hence we get $t \in G_a$. Thus there exists an $W \in \mathcal{B}$ such that $z + W \subset G_a$ for every $z \in G_a$ and so G_a is open.

Consequently, $\overleftarrow{a+U} \in \mathcal{N}_a$, when $U \in \mathcal{B}$. ∎

Acknowledgements

The authors would like to thanks to the referee for his/her helpful comments.

References

[1] S.M. Aseev, Quasilinear operators and their application in the theory of multivalued mappings, Proc. Steklov Inst. Math., 2(1986), 23-52.

[2] S. Çakan and Y. Yılmaz, Normed proper quasilinear spaces, J. Nonlinear Sci. Appl., 8(2015), 816-836.

[3] S. Çakan and Y. Yılmaz, On the quasimodules and normed quasimodules, Nonlinear Funct. Anal. Appl., 20(2) (2015), 269-288.

[4] S. Çakan and Y. Yılmaz, Lower and upper semi basis in quasilinear spaces, Erciyes University Journal of the Institute of Science and Technology, 31(2)(2015), 97-104.

[5] H. Bozkurt, S. Çakan and Y. Yılmaz, Quasilinear inner product spaces and Hilbert quasilinear spaces, International Journal of Analysis, Article ID 258389(2014), 7 pages.

[6] Y. Yılmaz, S. Çakan and Ş. Aytekin, Topological quasilinear spaces, Abstr. Appl. Anal., Article ID 951374(2012), 10 pages.

[7] R.E. Moore, R.B. Kearfott and M.J. Cloud, Introduction to Interval Analysis, SIAM, Philadelphia, USA, 2009.

[8] A. Wilansky, Modern Methods in Topological Vector Spaces, McGraw-Hill Int. Book Comp., New York, USA, 1978.
