

Research Article

Convergence Analysis for Double Splitting of Matrices

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Abstract: For single splittings of matrices, there are well-known convergence and comparison theorems. However, there are a few convergence theorems for double splitting. In this paper, we study this class of iterative methods. Furthermore, this paper gives new convergence results for double splitting of matrices.

Keywords: Convergence theorem; single splitting; double splitting; iterative methods; spectral radius.

1. Introduction

Consider the following linear system

$$Ax = b, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $b, x \in \mathbb{R}^n$.

Large families of iteration methods for solving (1) take the form

$$x^{i+1} = M^{-1}Nx^i + M^{-1}b, \quad i = 0, 1, \dots \quad (2)$$

where,

$$A = M - N.$$

The splitting of the coefficient matrix, where M is nonsingular, is called a single splitting of A [1]. There are many iterative methods based on single splitting; see, e.g., [2-10] and the references therein. The double splitting of A was introduced by Woznicki [1] as follows;

Splitting the matrix A in the form,

$$A = P - R + S, \quad (3)$$

called the double splitting of A , where P is a nonsingular matrix and we have the following iterative scheme spanned on three successive iterates,

$$x^{i+1} = P^{-1}Rx^i - P^{-1}Sx^{i-1} + P^{-1}b, \quad i = 0, 1, \dots \quad (4)$$

Following the idea of Golub and Varga [11], Woznicki wrote equation (4) in the following equivalent form,

$$\begin{pmatrix} x^{i+1} \\ x^i \end{pmatrix} = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix} \begin{pmatrix} x^i \\ x^{i-1} \end{pmatrix} + \begin{pmatrix} P^{-1}b \\ 0 \end{pmatrix}.$$

Then the iterative method given by (4) converges to the unique solution of (1) for all starting vectors x^0, x^1 if and only if the spectral radius of the iterative matrix,

$$W = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix}, \quad (5)$$

is less than unity, i.e., $\rho(W) < 1$.

When the double splitting defined by (3) has some special form, Jacobi double SOR, Gauss-Seidel double SOR and EWA double SOR methods are defined and their convergence are discussed in [1]. In [12], two double splittings and the corresponding two-step iterative methods are proposed for solving neutron diffusion equation. Recently, some convergence and comparison results for double splittings are also proved. Shen et al. in [13-14], presented convergence and some comparison theorems for double splittings of monotone matrices and of Hermitian positive definite matrices. Miao and Zheng [15] presented the comparison theorem for the spectral radii of matrices arising from double splittings of different monotone matrices. Zhang in [16], established some convergence results for double splittings of a non-Hermitian positive semi definite matrix. Song and Song [17] proved that if the double splitting is nonnegative it is convergent if and only if the corresponding single splitting is convergent. In this paper, we investigate double splitting of matrices. Furthermore, we establish the new convergence theorem for double splittings of different monotone matrices, which are different from the ones in [13-17].

2. Main Results

In this section we present our theorem.

Theorem: Let $A=P-R+S$ be double splitting of nonsingular matrix and let,

$$\begin{cases} P^t y_1 - S^t y_2 \geq 0 \Rightarrow (R^t - S^t) y_1 - S^t y_2 \geq 0, \\ A^t y_1 \geq 0 \Rightarrow (R^t - S^t) y_1 - S^t y_2 \geq 0, \\ y_2 \geq y_1 \Rightarrow y_1 \geq 0. \end{cases} \quad (6)$$

Where, t denotes the transpose. Then the spectral radius of the iterative matrix (5) is less than unity, i.e., $\rho(W) < 1$.

Proof: Let the conditions of (6) are true. Then we get,

$$M'Y \geq 0, \Rightarrow N'Y \geq 0, \quad (7)$$

$$H'Y \geq 0, \Rightarrow N'Y \geq 0, \quad (8)$$

where,

$$H = \begin{pmatrix} A & 0 \\ -I & I \end{pmatrix}, M = \begin{pmatrix} P & -S \\ 0 & I \end{pmatrix}, N = \begin{pmatrix} R-S & -S \\ I & 0 \end{pmatrix}, H = M - N, Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

and M and H are nonsingular. On the other hand, by duality theorems [18], we have the following relations;

(i) The relation (7) is equivalent to:

$$\text{for some nonnegative real matrix } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};$$

$$N = MX. \quad (9)$$

(ii) The relation (8) is equivalent to:

$$\text{for some nonnegative real matrix } Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix};$$

$$N = HZ. \quad (10)$$

It follows that;

$$X = M^{-1}N \geq 0, \quad Z = H^{-1}N \geq 0.$$

Let λ be an eigenvalue of $H^{-1}N$ with a corresponding eigenvector x .

Then,

$$Nx = \lambda Hx, \Rightarrow H^{-1}Nx = \lambda x.$$

So,

$$Nx = \frac{\lambda}{1+\lambda} Mx.$$

Note that $\lambda \neq -1$. Since if $\lambda = -1$, then for $x \neq 0$, we have;

$$Mx = \frac{1+\lambda}{\lambda} Nx = 0.$$

Which contradicts the nonsingularity on M . Hence, $\frac{\lambda}{1+\lambda}$ is an eigenvalue of $M^{-1}N$ with a corresponding eigenvector x .

Furthermore, let ζ be an eigenvalue of $M^{-1}N$ with a corresponding eigenvector t . Then,

$$Nt = \zeta Mt.$$

So,

$$Nt = \frac{\zeta}{1-\zeta} Ht.$$

Note that $\zeta \neq 1$. Since if $\zeta = 1$, then for $t \neq 0$, we have;

$$Ht = \frac{1-\zeta}{\zeta} Nt = 0.$$

Which contradicts the nonsingularity on H . Hence, $\frac{\zeta}{1-\zeta}$ is an eigenvalue of $H^{-1}N$ with a corresponding eigenvector t . Therefore, there is a one-to-one correspondence between the eigenvalues and eigenvectors of $H^{-1}N$ and $M^{-1}N$. Since both of these matrices are nonnegative by Pron-Frobenius theorem spectral radius $\rho(H^{-1}N)$ of $H^{-1}N$ is an eigenvalue of $H^{-1}N$ with a corresponding nonnegative nonvanishing eigenvector, and $\rho(M^{-1}N)$ is an eigenvalue of $M^{-1}N$ with a corresponding nonnegative nonvanishing eigenvector. Now, from [4, pp.96], we prove that $\rho(M^{-1}N)$ corresponds to a nonnegative eigenvalue of $H^{-1}N$. Since $\rho(M^{-1}N)$ is an eigenvalue of $M^{-1}N$, it corresponds to a real eigenvalue λ of $H^{-1}N$. We will show that if $\lambda < 0$ a contradiction ensues. We have;

$$\rho(M^{-1}N) = \frac{\lambda}{1+\lambda}, \quad H^{-1}Nx = \lambda x.$$

Since x is also an eigenvector of the nonnegative matrix $M^{-1}N$, $x \neq 0$ and $x \geq 0$, we have;

$$\lambda x = H^{-1}Nx = Zx.$$

which is the desired contradiction, since

$$\lambda < 0, x \geq 0, x \neq 0, Z \geq 0.$$

Hence ,

$$\lambda \geq 0.$$

And,

$$\rho(M^{-1}N) = \frac{\lambda}{1+\lambda}.$$

Again, since $H^{-1}N = Z \geq 0$, it follows that the largest nonnegative eigenvalue of $H^{-1}N$ is $\rho(H^{-1}N)$, and since $\frac{\lambda}{1+\lambda}$ is increasing for $\lambda \geq 0$, we have that;

$$\rho(M^{-1}N) = \frac{\rho(H^{-1}N)}{1+\rho(H^{-1}N)} < 1.$$

Where,

$$M^{-1}N = W = \begin{pmatrix} P^{-1}R & -P^{-1}S \\ I & 0 \end{pmatrix}.$$

Therefore, the spectral radius of the iterative matrix (5) is less than unity, i.e. $\rho(W) < 1$.

And the proof is completed.

3. Conclusion

In this paper, we studied double splitting technique for iterative methods. Furthermore, we have proposed the new convergence theorem for double splittings of different monotone matrices, which are different from the ones in [13-17].

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