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Research Article

Even Vertex Equitable Even Labeling of Path and Bistar Related Graphs

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Abstract: Let G be a graph with p vertices and q edges and $A = \{0, 2, 4, \cdots, q+1\}$ if q is odd or $A = \{0, 2, 4, \cdots, q\}$ if q is even. A graph G is said to be an even vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \cdots, 2q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits even vertex equitable even labeling is called an even vertex equitable even graph. In this paper, we prove that $P_n \odot mK_1$, $P_n(Q_m)$, $S^*(P_n \odot K_1)$, $S^*(L_n)$, $S^*(B_{n,n})$, $B^2_{n,n}$ and $S^{'}(B_{n,n})$ admit an even vertex equitable even labeling.

Keywords: Vertex equitable labeling; Even vertex equitable even labeling.

1 Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [2]. A labeling of a graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of vertices the labeling is called vertex labeling. If the domain is the set of edges, then we speak about edge labeling. If the labels are assigned to both vertices and edges then the labeling is called total labeling. For a dynamic survey of various graph labeling, we refer to Gallian [1]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. The concept of even vertex equitable even labeling was due to Lourdusamy *et al.* in [3]. In [3, 4, 5], Lourdusamy *et al.* proved that path, comb, complete bipartite, cycle, $K_2 + mK_1$, bistar, ladder, $S(L_n)$, $S(B_{n,n})$, $L_n \odot K_1$, P^2_n, $S(P_n \odot K_1)$, $S^{'}(P_n)$, $T(P_n)$, graph obtained by duplication of each vertex by an edge in P_n, Q_n, $S(Q_n)$, $D(Q_n)$, $A(T_n)$, $DA(T_n)$, $L_n \odot mK_1$, $C_n \odot K_1$, $P_n@P_m$, $\left\langle K_{1,n}^{(1)} \Delta K_{1,n}^{(2)} \right\rangle$, TL_n, $P_n \odot 2K_1$, $TL_n \odot K_1$, $T_n \odot K_1$, $Q_n \odot K_1$, $DA(Q_n) \odot K_1$, $DA(T_n) \odot K_1$, $S(DA(T_n))$, $S(DA(Q_n))$, $\langle B_{n,n} : w \rangle$ and $KY(m,n)$ admit an even vertex equitable even labeling. In [6, 7], Lourdusamy *et al.* proved that T_p-tree, $T\hat{o}P_n$, $T\hat{o}2P_n$, $T\hat{o}C_n(n \equiv 0, 3(mod\ 4))$, $T\tilde{o}C_n(n \equiv 0, 3(mod\ 4))$, $T\hat{o}K_{1,n}$, $T \odot \overline{K_n}$, $C_m \ominus P_n$, $C_n(Q_m)$, $\left[P_n; C_m^{(2)}\right]$, $C_m *_e C_n$ and the graph obtained by duplicating an arbitrary vertex and edge of a cycle C_n admit an even vertex equitable even labeling. Here, we prove that $P_n \odot mK_1$, $P_n(Q_m)$, $S^*(P_n \odot K_1)$, $S^*(L_n)$, $S^*(B_{n,n})$, $B_{n,n}^2$ and $S^{'}(B_{n,n})$ admit an even vertex equitable even labeling.

Definition 1.1 *Let G be a graph with p vertices and q edges and $A = \{0, 2, 4, \cdots, q+1\}$ if q is odd or $A = \{0, 2, 4, \cdots, q\}$ if q is even. A graph G is said to be an even vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \cdots, 2q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits even vertex equitable even labeling is called an even vertex equitable even graph.*

Definition 1.2 *The corona $G_1 \odot G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 1.3 *The graph $G = P_n(Q_m)$ is defined as isomorphic Quadrilateral snake attached with each vertex of path P_n, m is the number of C_4 attached in the Quadrilateral snake.*

Definition 1.4 *The super subdivision graph $S^*(G)$ is obtained from G by replacing every edge e of G by a complete bipartite graph $K_{2,m}, (m \geq 2)$ in such a way that the ends of e are merged with the two vertices of the 2-vertices part of $K_{2,m}$ after removing the edge e from G.*

Definition 1.5 *The graph $P_n \times P_2$ is called a ladder graph L_n.*

Definition 1.6 *The bistar $B_{n,n}$ is the graph obtained by attaching the apex vertices of two copies of $K_{1,n}$ by an edge.*

Definition 1.7 *For a simple connected graph G the square of graph G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.*

Definition 1.8 *For a graph G the splitting graph $S^{'}(G)$ of a graph G is obtained by adding a new vertex $v^{'}$ corresponding to each vertex v of G such that $N(v) = N(v^{'})$.*

2 Main Results

Theorem 2.1 *The graph $P_n \odot mK_1$ is an even vertex equitable even graph for $n \equiv 0\ (mod\ 2)$.*

Proof: Let $G = P_n \odot mK_1$. Let $V(G) = \{u_i : 1 \leq i \leq n\} \bigcup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_i u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$. Then, G is of order $mn + n$ and size $mn + n - 1$.
Define $f : V(G) \to A = \{0, 2, \cdots, mn + n\}$ as follows:
For $1 \leq i \leq n$,

$$f(u_i) = \begin{cases} (m+1)(i-1) & \text{if } i \text{ is odd} \\ (m+1)i & \text{if } i \text{ is even ;} \end{cases}$$

$$f(u_{ij}) = \begin{cases} (m+1)(i-1) + 2j & \text{if } i \text{ is odd and } 1 \leq j \leq m \\ (m+1)(i-2) + 2j & \text{if } i \text{ is even and } 1 \leq j \leq m. \end{cases}$$

It can be verified that the induced edge labels of $P_n \odot mK_1$ are $2, 4, \cdots, 2mn+2n-2$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $P_n \odot mK_1$ is an even vertex equitable even graph.

Example 2.2 *An even vertex equitable even labeling of $P_4 \odot 4K_1$ is shown in Figure 2.1.*

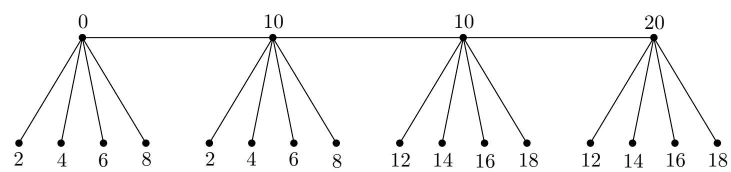

Figure 2.1

Theorem 2.3 *The graph $P_n(Q_m)$ is an even vertex equitable even graph.*

Proof: Let $G = P_n(Q_m)$. Let $V(G) = \{u_i : 1 \leq i \leq n\} \bigcup \{u_{ij}, v_{ij}, w_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_i v_{i1}, u_i w_{i1} : 1 \leq i \leq n\} \bigcup \{v_{ij} u_{ij}, w_{ij} u_{ij} : 1 \leq i \leq n; 1 \leq j \leq m\} \bigcup \{u_{ij} v_{ij+1}, u_{ij} w_{ij+1} : 1 \leq i \leq n; 1 \leq j \leq m-1\}$. Then, G is of order $3mn + n$ and size $4mn + n - 1$.

$$\text{Define } f : V(G) \to A = \begin{cases} 0, 2, \cdots, 4mn + n & \text{if } 4mn + n - 1 \text{ is odd} \\ 0, 2, \cdots, 4mn + n - 1 & \text{if } 4mn + n - 1 \text{ is even} \end{cases}$$

as follows:

$$f(u_i) = \begin{cases} (4m+1)(i-1) & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ (4m+1)i & \text{if } i \text{ is even and } 1 \leq i \leq n \text{ ;} \end{cases}$$

For $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$f(u_{ij}) = \begin{cases} (4m+1)(i-1) + 4j & \text{if } i \text{ is odd} \\ (4m+1)i - 4j & \text{if } i \text{ is even ;} \end{cases}$$

$$f(v_{ij}) = \begin{cases} (4m+1)(i-1) + 4(j-1) + 2 & \text{if } i \text{ is odd} \\ (4m+1)i - 4j & \text{if } i \text{ is even ;} \end{cases}$$

$$f(w_{ij}) = \begin{cases} (4m+1)(i-1)+4j & \text{if } i \text{ is odd} \\ (4m+1)i-4j+2 & \text{if } i \text{ is even.} \end{cases}$$

It can be verified that the induced edge labels of $P_n(Q_m)$ are $2, 4, \cdots, 8nm+2n-2$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $P_n(Q_m)$ is an even vertex equitable even graph.

Example 2.4 *An even vertex equitable even labeling of* $P_3(Q_3)$ *is shown in Figure* 2.2.

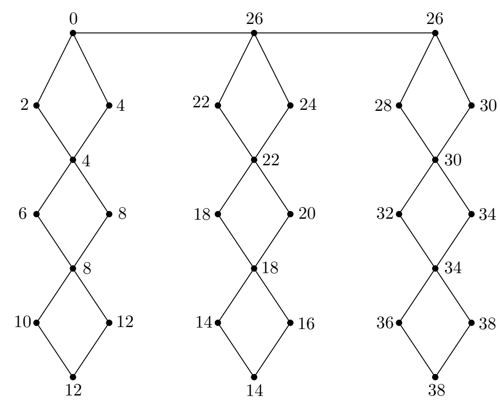

Figure 2.2

Theorem 2.5 *The super subdivision graph* $S^*(P_n \odot K_1)$ *is an even vertex equitable even graph.*

Proof: Let $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ be the vertices of $P_n \odot K_1$. The super subdivision graph $S^*(P_n \odot K_1)$ is obtained by replacing each edge of $P_n \odot K_1$ is replaced by a complete bipartite graph $K_{2,m}$. Let $V(S^*(P_n \odot K_1)) = \{u_i, v_i, v_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \bigcup \{u_{ij} : 1 \leq i \leq n-1, 1 \leq j \leq m\}$ and $E(S^*(P_n \odot K_1)) = \{u_i v_{ij}, v_{ij} v_i : 1 \leq i \leq n, 1 \leq j \leq m\} \bigcup \{u_i u_{ij}, u_{ij} u_{i+1} : 1 \leq i \leq n-1, 1 \leq j \leq m\}$. Then $S^*(P_n \odot K_1)$ is of order $2n + 2mn - m$ and size $4mn - 2m$.
Define $f : V(S^*(P_n \odot K_1)) \to \{0, 2, \cdots, 4mn - 2m\}$ as follows:

$$f(u_i) = \begin{cases} 2m(2i-1) & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ 4m(i-1) & \text{if } i \text{ is even and } 1 \leq i \leq n \ ; \end{cases}$$

$$f(v_i) = \begin{cases} 4m(i-1) & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ 2m(2i-1) & \text{if } i \text{ is even and } 1 \leq i \leq n \ ; \end{cases}$$

For $1 \leq j \leq m$,

$$f(u_{ij}) = \begin{cases} 2m(2i-1)+2j & \text{if } i \text{ is odd and } 1 \leq i \leq n-1 \\ 4mi+2j & \text{if } i \text{ is even and } 1 \leq i \leq n-1 \ ; \end{cases}$$

$$f(v_{ij}) = \begin{cases} 2j & \text{if } i = 1 \\ 2m(2i-1)-2(j-1) & \text{if } i \text{ is even and } 2 \leq i \leq n \\ 4m(i-1)-2(j-1) & \text{if } i \text{ is odd and } 2 \leq i \leq n. \end{cases}$$

It can be verified that the induced edge labels of $S^*(P_n \odot K_1)$ are $2, 4, \cdots, 8mn - 4m$

and $|v_f(a)-v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $S^*(P_n \odot K_1)$ is an even vertex equitable even graph.

Example 2.6 *An even vertex equitable even labeling of $S^*(P_3 \odot K_1)$ is shown in Figure 2.3.*

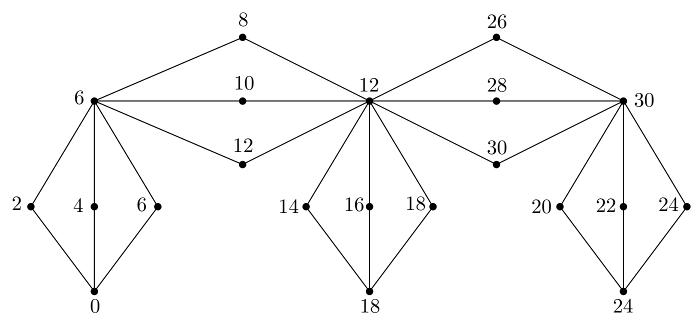

Figure 2.3

Theorem 2.7 *The super subdivision graph $S^*(L_n)$ is an even vertex equitable even graph.*

Proof: Let $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ be the vertices of L_n. The super subdivision graph $S^*(L_n)$ is obtained by replacing each edge of L_n is replaced by a complete bipartite graph $K_{2,m}$. Let $V(S^*(L_n)) = \{u_i, v_i, w_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \bigcup \{u_{ij}, v_{ij} : 1 \leq i \leq n-1, 1 \leq j \leq m\}$ and $E(S^*(L_n)) = \{u_i w_{ij}, v_i w_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \bigcup \{u_i u_{ij}, u_{ij} u_{i+1}, v_i v_{ij}, v_{ij} v_{i+1} : 1 \leq i \leq n-1, 1 \leq j \leq m\}$. Then $S^*(L_n)$ is of order $2nm + 2n - m$ and size $6mn - 4m$.
Define $f : V(S^*(L_n)) \to \{0, 2, \cdots, 6mn - 4m\}$ as follows:

$$f(u_i) = \begin{cases} 6m(i-1) & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ 2m(3i-2) & \text{if } i \text{ is even and } 1 \leq i \leq n \text{ ;} \end{cases}$$

$$f(v_i) = \begin{cases} 2m(3i-2) & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ 6m(i-1) & \text{if } i \text{ is even and } 1 \leq i \leq n \text{ ;} \end{cases}$$

For $1 \leq j \leq m$,

$$f(u_{ij}) = \begin{cases} 6mi + 2j & \text{if } i \text{ is odd and } 1 \leq i \leq n-1 \\ 2m(3i-2) + 2j & \text{if } i \text{ is even and } 1 \leq i \leq n-1 \text{ ;} \end{cases}$$

$$f(v_{ij}) = \begin{cases} 2m(3i-2) + 2j & \text{if } i \text{ is odd and } 1 \leq i \leq n-1 \\ 6mi + 2j & \text{if } i \text{ is even and } 1 \leq i \leq n-1 \text{ ;} \end{cases}$$

$$f(w_{ij}) = \begin{cases} 2j & \text{if } i = 1 \\ 6m(i-1) - 2(j-1) & \text{if } 2 \leq i \leq n. \end{cases}$$

It can be verified that the induced edge labels of $S^*(L_n)$ are $2, 4, \cdots, 12mn - 8m$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $S^*(L_n)$ is an even vertex equitable even graph.

Example 2.8 *An even vertex equitable even labeling of $S^*(L_3)$ is shown in Figure 2.4.*

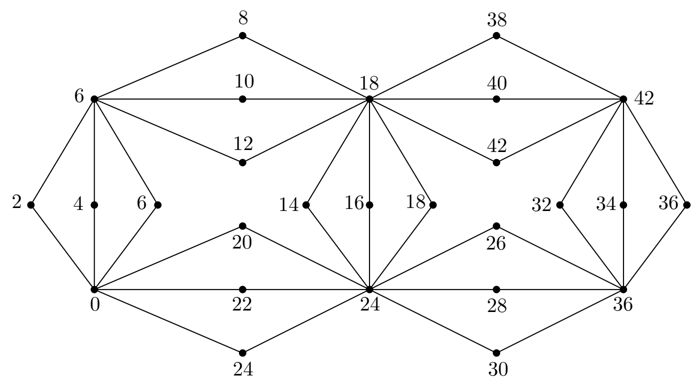

Figure 2.4

Theorem 2.9 *The super subdivision graph $S^*(B_{n,n})$ is an even vertex equitable even graph.*

Proof: Let $u, u_1, u_2, \cdots, u_n, v, v_1, v_2, \cdots, v_n$ be the vertices of the bistar $B_{n,n}$. The super subdivision graph $S^*(B_{n,n})$ is obtained by replacing each edge of $B_{n,n}$ is replaced by a complete bipartite graph $K_{2,m}$. Let $V(S^*(B_{n,n})) = \{u_i, v_i, w_j, u_{ij}, v_{ij} : 1 \leq i \leq n,\ 1 \leq j \leq m\} \bigcup \{u, v\}$ and $E(S^*(B_{n,n})) = \{u_iu_{ij}, u_{ij}u, uw_j, w_jv, vv_{ij}, v_iv_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$. Then $S^*(B_{n,n})$ is of order $2mn+m+2n+2$ and size $4mn+2m$. Define $f : V(S^*(B_{n,n})) \to \{0, 2, \cdots, 4mn + 2m\}$ as follows:

$f(v) = 2m;$

$f(u) = 4nm;$

$f(v_i) = 4m(i-1),\ 1 \leq i \leq n;$

$f(u_i) = 2m(2i+1),\ 1 \leq i \leq n;$

For $1 \leq j \leq m$,

$f(w_j) = 4mn - 2(j-1);$

$$f(v_{ij}) = \begin{cases} 2j & \text{if } \ i = 1 \\ 4m(i-1) - 2(j-1) & \text{if } \ 2 \leq i \leq n \ ; \end{cases}$$

$f(u_{ij}) = 2m(2i+1) - 2(j-1),\ 1 \leq i \leq n.$

It can be verified that the induced edge labels of $S^*(B_{n,n})$ are $2, 4, \cdots, 8mn + 4m$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $S^*(B_{n,n})$ is an even vertex equitable even graph.

Example 2.10 *An even vertex equitable even labeling of $S^*(B_{2,2})$ is shown in Figure* 2.5.

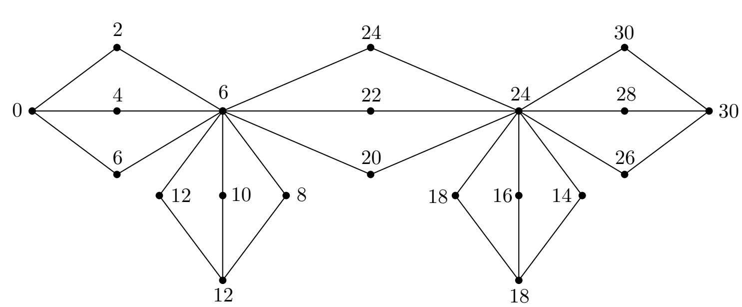

Figure 2.5

Theorem 2.11 *The graph $B^2_{n,n}$ is an even vertex equitable even graph.*

Proof: Let $u, u_1, u_2, \cdots, u_n, v, v_1, v_2, \cdots, v_n$ be the vertices of the bistar $B_{n,n}$. Let $G = B^2_{n,n}$ with $V(G) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{uv\} \bigcup \{uu_i, vv_i, uv_i, vu_i : 1 \leq i \leq n\}$. Then $V(G) = 2n + 2$ and $E(G) = 4n + 1$.
Define $f : V(G) \to \{0, 2, \cdots, 4n + 2\}$ as follows:

$f(u) = 0;$
$f(v) = 4n + 2;$
$f(u_i) = 2i, \ 1 \leq i \leq n;$
$f(v_i) = 2n + 2i, \ 1 \leq i \leq n.$

It can be verified that the induced edge labels of $B^2_{n,n}$ are $2, 4, \cdots, 8n + 2$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $B^2_{n,n}$ is an even vertex equitable even graph.

Example 2.12 *An even vertex equitable even labeling of $B^2_{4,4}$ is shown in Figure* 2.6.

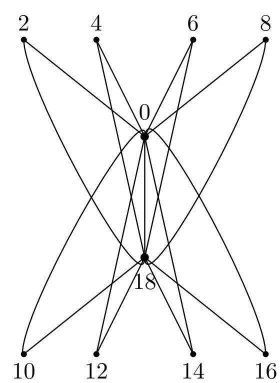

Figure 2.6

Theorem 2.13 *The graph $S^{'}(B_{n,n})$ is an even vertex equitable even graph.*

Proof: Let $u, u_1, u_2, \cdots, u_n, v, v_1, v_2, \cdots, v_n$ be the vertices of the bistar $B_{n,n}$. Let $G = S^{'}(B_{n,n})$ with $V(G) = \{u_i, v_i, u^{'}_i, v^{'}_i : 1 \leq i \leq n\} \bigcup \{u, v, u^{'}, v^{'}\}$ and $E(G) = \{uv, uv^{'}, vu^{'}\} \bigcup \{uu^{'}_i, uu_i, u^{'}u_i, vv^{'}_i, vv_i, v^{'}v_i, : 1 \leq i \leq n\}$. Then $V(G) = 4n + 4$ and $E(G) = 6n + 3$.
Define $f : V(G) \to \{0, 2, \cdots, 6n + 4\}$ as follows:

$f(u) = 0;$
$f(v) = 6n + 4;$
$f(u^{'}) = 2n + 2;$
$f(v^{'}) = 4n + 2;$
$f(u_i) = 2n + 2i, \ 1 \leq i \leq n;$
$f(u^{'}_i) = 2i, \ 1 \leq i \leq n;$
$f(v_i) = 2n + 2 + 2i, \ 1 \leq i \leq n;$

$f(v_i^{'}) = 4n + 2 + 2i, \ 1 \leq i \leq n.$

It can be verified that the induced edge labels of $S^{'}(B_{n,n})$ are $2, 4, \cdots, 12n+6$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Thus, $S^{'}(B_{n,n})$ is an even vertex equitable even graph.

Example 2.14 *An even vertex equitable even labeling of* $S^{'}(B_{5,5})$ *is shown in Figure* 2.7.

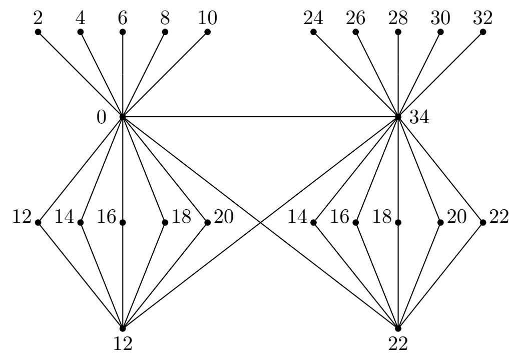

Figure 2.7

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