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Research Article

Square Root of an Operator: Laplacian & Weyl and Dirac Equations

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Abstract: With the analysis of the Cauchy-Riemann conditions for a holomorphic function we obtain the square root of the two dimensional Laplacian operator. Also, we illustrate by means of the Pauli matrices how to linearize (second order operators) $^{1/2}$ associated to the Weyl and Dirac equations.

Keywords: Cauchy-Riemann conditions, Pauli matrices, Laplacian operator, Weyl and Dirac equations

1. Introduction

In complex variable, the holomorphic character of $f(z) = u + i\, v$ implies the Cauchy [1]-Riemann [2] conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ , \tag{1}$$

as obtained by D'Alembert [3], Euler [4], Lagrange [5] and Hamilton [6]. Such conditions are equivalent to [7, 8]:

$$\left(\frac{\partial}{\partial x} + i\ \frac{\partial}{\partial y}\right)(u + iv) = 0\ , \tag{2}$$

which are important to generalize (1) to the quaternionic case [9].

Moreover, there exists a matricial version of (1) [10], which is:

$$\hat{O}\binom{u}{v} = \vec{0} \quad , \quad \hat{O} = \begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \end{pmatrix}, \tag{3}$$

So, it is immediate to realize that for an arbitrary vector $\vec{w}$:

$$\hat{O}^2\vec{w} = \nabla^2\vec{w} \qquad \therefore \qquad \sqrt{\nabla^2} = \hat{O} \ , \tag{4}$$

then, $\hat{O}$ is the square root of the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, that is, it was possible to linearize the operator $(\nabla^2)^{1/2}$.

The Pauli matrices [11, 12]:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{5}$$

first obtained by Cayley [13] and Sylvester [14], were used [11, 15] in the Schrödinger equation to incorporate the electron spin. So the operator (3) adopts the form $\hat{O} = \sigma_1 \frac{\partial}{\partial x} + \sigma_3 \frac{\partial}{\partial y}$, and in quantum mechanics the Laplacian operator is related to the square of the linear momentum operator, then from (4) it is natural to think that operators of the type (quadratic in $\hat{p})^{1/2}$ can be linearized in quantum physics. In Sections 2 and 3 we apply this idea to motivate the Weyl [16] and Dirac [17] equations, respectively.

2. The Weyl Equations

With the matrices (5) it is easy to see that:

$$\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3 = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} , \tag{6}$$

or, after squaring,$(\vec{\sigma} \bullet \vec{p})^2 = I_{2x2} \, ({p_1}^2 + {p_2}^2 + {p_3}^2)$, then it is natural to propose the linearization as follows:

$$\vec{\sigma} \bullet \vec{p} = \pm I \sqrt{{p_1}^2 + {p_2}^2 + {p_3}^2} \ , \tag{7}$$

with p_r as quantum mechanical operators. A relativistic free particle with zero rest mass has energy $E = c \, ({p_1}^2 + {p_2}^2 + {p_3}^2)^{1/2}$ and due to (7) [$\tilde{h}$ = Planck constant /2π]:

$$c \, (\vec{\sigma} \bullet \vec{p}) = \pm i \, \tilde{h} \, I \, \frac{\partial}{\partial t} \ , \tag{8}$$

that for each sign operates on a two- component column vector as:

$$c \, (\vec{\sigma} \bullet \vec{p})\phi_R = i \, \tilde{h} \, \frac{\partial}{\partial t}\phi_R \qquad , \qquad c \, (\vec{\sigma} \bullet \vec{p})\phi_L = -i \, \tilde{h} \, \frac{\partial}{\partial t}\phi_L \tag{9}$$

which are the equations proposed by Weyl [16, 18] for the neutrino, spin 1/2. Next Section shows how the Dirac equation for a free particle of different from zero mass leads to (9).

3. The Dirac Equation

With the 4x4 matrices [19, 20]:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{10}$$

one has

$$m_0 c\, \beta + \vec{\alpha} \bullet \vec{p} = \begin{pmatrix} m_0 c\, I & \vec{\sigma} \bullet \vec{p} \\ \vec{\sigma} \bullet \vec{p} & -m_0 c\, I \end{pmatrix},$$

or $(m_0 c\, \beta + \vec{\alpha} \bullet \vec{p})^2 = ({m_0}^2 c^2 + {p_1}^2 + {p_2}^2 + {p_3}^2)\, I_{4x4}$, which generalizes (7):

$$m_0 c\, \beta + \vec{\alpha} \bullet \vec{p} = I \sqrt{{m_0}^2 c^2 + {p_1}^2 + {p_2}^2 + {p_3}^2} \ , \tag{11}$$

this in turn, together with $E = c\sqrt{{m_0}^2 c^2 + p^2} = i\, \tilde{h}\, \frac{\partial}{\partial t}$, generates the Dirac equation [17, 19, 21-23] for a free electron, spin 1/2 [24]:

$$(m_0 c^2 \beta + c\vec{\alpha} \bullet \vec{p})\, \psi_{4x1} = i\, \tilde{h}\, \frac{\partial}{\partial t}\, \psi_{4x1} \, . \tag{12}$$

When m_0= 0, equation (12) acquires the form:

$$\begin{pmatrix} 0 & c\, \vec{\sigma} \bullet \vec{p} \\ c\, \vec{\sigma} \bullet \vec{p} & 0 \end{pmatrix} \psi = i\, \tilde{h}\, \frac{\partial \psi}{\partial t} \ , \qquad \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{13}$$

that is

$$c\, (\vec{\sigma} \bullet \vec{p})\xi = i\, \tilde{h}\, \frac{\partial \eta}{\partial t} \qquad\qquad c\, (\vec{\sigma} \bullet \vec{p})\eta = i\, \tilde{h}\, \frac{\partial \xi}{\partial t} \ .$$

By adding and subtracting these equations we get the Weyl equations (9), with $\phi_R = (\xi + \eta)/\sqrt{2}$ y $\phi_L = (\xi - \eta)/\sqrt{2}$.

The expressions (4), (7) and (11) exhibit linear forms of the square root of important operators of Mathematical Physics, and from here it follows the relevant role of the Pauli matrices.

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