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Research Article

Super Mean Labelings of Cyclic Snakes

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Abstract:A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge uv depending on the vertex labels. A *super mean labeling* f is an injection from V to the set $\{1, 2, ..., p + q\}$ that induces for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the set of all vertex labels and the induced edge labels is $\{1, 2, ..., p + q\}$. In this paper we present super mean labeling of kC_n-snakes and kC-snakes.

Keywords: Mean labeling, super mean labeling, cyclic snakes.

1 Introduction

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge uv depending on the vertex labels. Let $G = (V, E)$ be a simple graph with p vertices and q edges. A *mean labeling* f is an injection from V to the set $\{0, 1, 2, ..., q\}$ that induces for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the set of edge labels is $\{1, 2, ..., q\}$. Mean labeling was introduced by Somasundaram and Ponraj [7]. A graph that accepts a mean labeling is known as mean graph. A *super mean labeling* f is an injection from V to the set $\{1, 2, ..., p + q\}$ that induces for each edge uv the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ such that the set of all vertex labels and the induced edge labels is $\{1, 2, ..., p + q\}$. Super mean labeling was introduced by Ponraj and Ramya [5]. A graph that accepts a super mean labeling is known as super mean graph. Some results on mean labeling and super mean labeling are given in [3], [4], [6], and [8]. For a summary on various graph labeling see the Dynamic survey of graph labeling by Gallian [2].

For given $k \geq 3$ and $n \geq 3$, a kC_n -snake has been defined as a connected graph in which all the k many blocks are isomorphic to the cycle C_n and the block-cut point graph is a path. Let P be the path in G of minimum length that contains all the cut vertices of a kC_n-snake. Barrientos [1] has proved that any kC_n-snake is represented by a string, $s_1, s_2, s_3, ...s_{k-2}$ of integers where the i^{th} integer, $s_i \in S_n := \left\{1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor\right\}$, on the string is the distance between the i^{th} and the $(i+1)^{th}$ cut vertices on the path P. This representation is not unique, because it depends on the direction in which we walk through P. This problem can be avoided by considering the two possible strings obtained as the same. For example, the string of an $8C_5$-snake shown in Figure 2 is $2, 1, 1, 2, 2, 1$. A kC_n -snake is said to be linear if each integer of its string is $\left\lfloor \frac{n}{2} \right\rfloor$. Vasuki and Nagarajan [8] proved that linear kC_n-snakes for $n \geq 3$ and $n \neq 4$ are super mean graphs. Here in this paper we prove that kC_{2n+1}-snakes for all $n \geq 1$ and kC_{2n}-snakes with the string $s_1, s_2, s_3, ...s_{k-2}$ where $s_i \geq 2$ for each i are super mean graphs for all $k \geq 1$ and $n \geq 3$. We also define a kC-snake and find the conditions under which they are super mean graphs.

2 Main Results

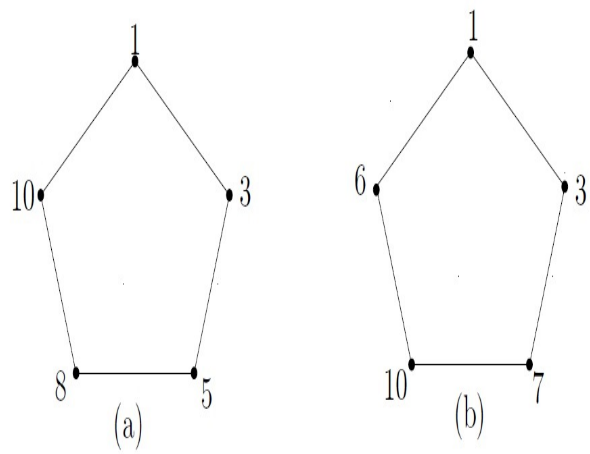


Figure 1: *type* 1 and *type* 2 labeling of C_5

Consider the labeling of C_5 in Figure 1. We call the labeling in (a) as *type* 1 and that in (b) as *type* 2. In both types of labeling the set of all vertex labels and the induced edge labels of C_5 is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. It is also possible to add a fixed *offset* x to any label so that the set of all vertex labels and the induced edge labels of C_5 is $\{x+1, x+2, x+3, x+4, x+5, x+6, x+7, x+8, x+9, x+10\}$. These two types of labeling are used in the following theorem.

Theorem 2.1. *Any kC_5-snake is a super mean graph.*

Proof. Let G be any kC_5-snake. Then its string is of the form s_1, s_2, ..., s_{k-2}, where s$_i \in \{1, 2\}$ for each i. Let $B_1, B_2, B_3, ..., B_k$ be the consecutive blocks of G.

First we label the blocks as given in Table 1. Then B_{i+1} and B_i are connected by

Blocks	Type used	offset x
B_1	either type	0
$B_{i+1}, 1 \le i \le k-2$	*type* s_i	$9i$
B_k	either type	$9k-9$

Table 1:

identifying their vertices of common label $9i+1$. That is, the biggest label of B_i and the smallest label of B_{i+1}. Observe that the set of all vertex labels and the induced edge labels of the block B_i is $\{9(i-1)+1, 9(i-1)+2, ..., 9(i-1)+10\}$. It can be easily verified that the set of all vertex labels and the induced edge labels of G is $\{1, 2, 3, ..., 9k+1\}$. Thus we have obtained a super mean labeling of G. A super mean labeling of an $8C_5$-snake is shown in Figure 2. □

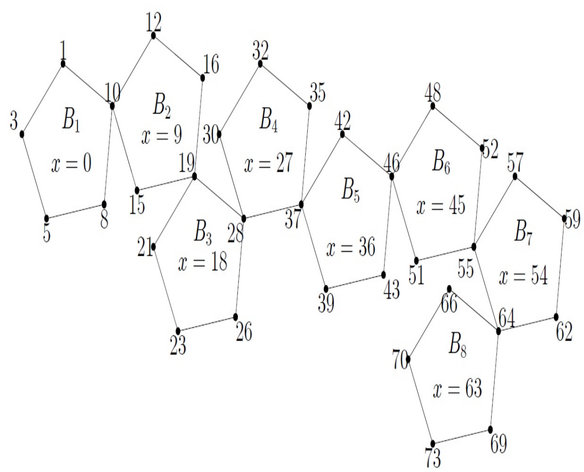


Figure 2: A super mean labeling of an $8C_5$-snake

Next we proceed to prove the general case by introducing a notation.

Notation 2.2. *We denote a labeling* $h : V(C_m) \longrightarrow \{1, 2, ..., 2m\}$ *that places the two labels* 1 *and* $2m$ *on two vertices with a distance* d *from each other such that the set of all vertex labels and the induced edge labels of* C_m *is* $\{1, 2, ..., 2m\}$ *as a type* d *labeling. It is also possible to add a fixed offset* x *to any label so that the set of all vertex labels and the induced edge labels of* C_m *is* $\{x+1, x+2, ..., x+2m\}$. *This is illustrated in Figure 3. For each edge* $uv \in E(C_m)$, *let* $h^*(uv)$ *denote the induced edge label. For* $V_1 \subseteq V(C_m)$ *and* $E_1 \subseteq E(C_m)$, *we define* $h(V_1) = \{h(v) \ / v \in V_1\}$ *and* $h^*(E_1) = \{h^*(e) \ / e \in E_1\}$. *Then* $h(V(C_m))$ *and* $h^*(E(C_m))$ *denote respectively the set of vertex labels and the set of induced edge labels.*

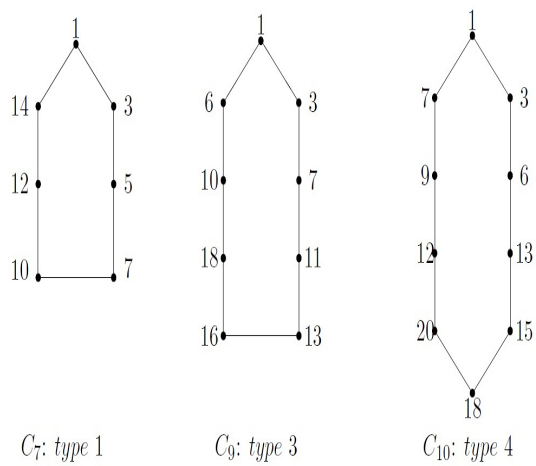


Figure 3: *type d* labeling of C_m

Let C_{2n+1} be a cycle with consecutive vertices $v_0, v_1, v_2, ..., v_{2n}$ and $s \in S_{2n+1}$. Consider the labeling $g_s : V(C_{2n+1}) \longrightarrow \{1, 2, ..., 4n+2\}$ defined as follows:

$$g_1(v_j) = \begin{cases} 1 & : j = 0 \\ 4n - 2j + 4 & : 1 \leq j \leq n \\ 4n - 2j + 3 & : n+1 \leq j \leq 2n \end{cases},$$

for $2 \leq s \leq n-1$

$$g_s(v_j) = \begin{cases} 1 & : j = 0 \\ 4j + 2 & : 1 \leq j \leq s-1 \\ 4n + 2s - 2j + 2 & : s \leq j \leq n \\ 4n + 2s - 2j + 1 & : n+1 \leq j \leq 2n - s + 1 \\ 8n - 4j + 3 & : 2n - s + 2 \leq j \leq 2n \end{cases},$$

$$g_n(v_j) = \begin{cases} 1 & : j = 0 \\ 4j + 2 & : 1 \leq j \leq n \\ 8n - 4j + 3 & : n+1 \leq j \leq 2n \end{cases}.$$

Now we prove that g_s is a *type* s labeling.

Case 1: $s = 1$

Let

$$V_{11} = \{v_j : \ 1 \leq j \ \leq n\},$$

$$V_{21} = \{v_j : \ n+1 \leq j \ \leq 2n\} \cup \{v_0\},$$

$$E_{11} = \{v_j v_{j+1} : \ 1 \leq j \ \leq n \ \},$$
$$E_{21} = \{v_j v_{j+1} : \ n+1 \leq j \ \leq 2n-1 \ \} \cup \{v_{2n}v_0, v_0v_1\},$$
$$A_{k1} = g_1(V_{k1}) \text{ and } B_{k1} = g_1^*(E_{k1}), k = 1, 2.$$

Then

$$A_{11} = 4n - 2j + 4 : \ 1 \leq j \ \leq n\}$$
$$= \{4n+2, x+4n, ..., x+2n+6, x+2n+4\}$$
$$= \{2n+4, x+2n+6, ..., x+4n, x+4n+2\},$$
$$A_{21} = \{4n - 2j + 3 / \ n+1 \leq j \ \leq 2n+1\}$$
$$= \{2n+1, 2n-1, ..., 3, 1\}$$
$$= \{1, 3..., 2n-1, 2n+1\},$$
$$B_{11} = \{4n - 2j + 3 : \ 1 \leq j \ \leq n \ \}$$
$$= \{4n+1, 4n-1, ..., 2n+5, 2n+3\}$$
$$= \{2n+3, 2n+5, ..., 4n-1, 4n+1\},$$
$$B_{21} = \{4n - 2j + 4 : \ n+1 \leq j \ \leq 2n+1\}$$
$$= \{2n+2, 2n, ..., 4, 2\}$$
$$= \{2, 4, ..., 2n, 2n+2\}.$$

The sets $V_{i1}, \ i = 1, 2,$ form a partition of $V(C_{2n+1})$, the sets $E_{i1}, \ i = 1, 2,$ form a partition of $E(C_{2n+1})$ and

$$g_1(V(C_{2n+1})) \cup g_1^*(E(C_{2n+1})) = A_{11} \cup A_{21} \cup B_{11} \cup B_{21}$$
$$= \{1, 2, ..., 4n+2\}.$$

It is easy to verify that g_1 is of *type 1*
Case 2 : $2 \leq s \leq n-1$.
Let

$$V_{1s} = \{v_0\},$$
$$V_{2s} = \{v_j : \ 1 \leq j \ \leq s-1\},$$
$$V_{3s} = \{v_j : \ s \leq j \ \leq n\},$$
$$V_{4s} = \{v_j : \ n+1 \leq j \ \leq 2n-s+1\},$$
$$V_{5s} = \{v_j : \ 2n-s+2 \leq j \ \leq 2n\},$$
$$E_{1s} = \{v_{2n}v_0\},$$
$$E_{2s} = \{v_j v_{j+1} : \ 0 \leq j \ \leq s-2 \ \},$$
$$E_{3s} = \{v_{s-1}v_s\},$$

$$E_{4s} = \{v_j v_{j+1} :\ s \le j\ \le n-1\ \},$$
$$E_{5s} = \{v_n v_{n+1}\},$$
$$E_{6s} = \{v_j v_{j+1} :\ n+1 \le j\ \le 2n-s\ \},$$
$$E_{7s} = \{v_j v_{j+1} :\ 2n-s+1 \le j\ \le 2n-1\ \},$$
$$A_{ks} = g_s(V_{ks}), 1 \le k \le 5 \ and\ B_{ks} = g_s^*(E_{ks}), 1 \le k \le 7.$$

Then

$$A_{1s} = \{1\},$$
$$A_{2s} = \{4j+2 :\ 1 \le j\ \le s-1\}$$
$$= \{6, 10, ...4s-6, 4s-2\},$$
$$A_{3s} = \{4n+2s-2j+2 :\ s \le j\ \le n\}$$
$$= \{2n+2s+2, 2n+2s+4, ..., 4n, 4n+2\},$$
$$A_{4s} = \{4n+2s-2j+1 :\ n+1 \le j\ \le 2n-s+1\}$$
$$= \{4s-1, 4s+1, ..., 2n+2s-3, 2n+2s-1\},$$
$$A_{5s} = \{8n-4j+3 :\ 2n-s+2 \le j\ \le 2n\}$$
$$= \{3, 7, ...4s-9, 4s-5\},$$
$$B_{1s} = \{2\},$$
$$B_{2s} = \{4j+4 :\ 0 \le j\ \le s-2\ \}$$
$$= \{4, 8, ...4s-8, 4s-4\},$$
$$B_{3s} = \{2n+2s\},$$
$$B_{4s} = \{4n+2s-2j+1 :\ s \le j\ \le n-1\ \}$$
$$= \{2n+2s+3, 2n+2s+5, ..., 4n-1, 4n+1\},$$
$$B_{5s} = \{2n+2s+1\},$$
$$B_{6s} = \{4n+2s-2j :\ n+1 \le j\ \le 2n-s\ \}$$
$$= \{4s, 4s+2, ..., 2n+2s-4, 2n+2s-2\},$$
$$B_{7s} = \{8n-4j+1 :\ 2n-s+1 \le j\ \le 2n-1\ \}$$
$$= \{5, 9, ...4s-7, 4s-3\},$$
$$A_{1s} \cup B_{1s} = \{1, 2\},$$
$$A_{5s} \cup B_{2s} \cup B_{7s} \cup A_{2s} = \{3, 4, ..., 4s-2\},$$
$$A_{4s} \cup B_{6s} \cup B_{3s} = \{4s-1, 4s, ..., 2n+2s-1, 2n+2s\},$$
$$B_{5s} \cup B_{4s} \cup A_{3s} = \{2n+2s+1, 2n+2s+2, ..., 4n+2\}.$$

The sets $V_{is},\ i=1,2,3,4,5,$ form a partition of $V(C_{2n+1})$, the sets $E_{is},\ i=1,2,3,4,5,6,7,$ form a partition of $E(C_{2n+1})$ and

$$\begin{aligned} g_s\left(V(C_{2n+1})\right)\cup g_s^*\left(E(C_{2n+1})\right) &= \left(\bigcup_{i=1}^{5}A_{is}\right)\cup\left(\bigcup_{i=1}^{7}B_{is}\right) \\ &= \{1,2,...,4n+2\}. \end{aligned}$$

It is easy to verify that $g_s, 2\le s\le n-1$ is of *type* s.

Case 3: $s=n$

Let

$$\begin{aligned} &V_{1n}=\{v_0\} \\ &V_{2n}=\{v_j/\ 1\le j\ \le n\} \\ &V_{3n}=\{v_j/\ n+1\le j\ \le 2n\} \\ &E_{1n}=\{v_{2n}v_0\} \\ &E_{2n}=\{v_jv_{j+1}/\ 0\le j\ \le n-1\ \} \\ &E_{3n}=\{v_jv_{j+1}/\ n\le j\ \le 2n-1\ \} \\ &A_{kn}=g_n(V_{kn}) \text{ and } B_{kn}=g_n^*(E_{kn})\ ,k=1,2,3 \end{aligned}$$

Then

$$\begin{aligned} A_{1n} &= \{1\}, \\ A_{2n} &= \{4j+2:\ 1\le j\ \le n\} \\ &= \{6,10,...,4n,4n+2\}, \\ A_{3n} &= \{8n-4j+3:\ n+1\le j\ \le 2n\} \\ &= \{3,7,...,4n-5,4n-1\}, \\ B_{1n} &= \{2\}, \\ B_{2n} &= \{4j+4:\ 0\le j\ \le n-1\ \} \\ &= \{4,8,..,4n-4,4n\}, \\ B_{3n} &= \{8n-4j+1:\ n\le j\ \le 2n-1 \\ &= \{5,9,...,4n-3,4n+1\}, \\ A_{1n}\cup A_{3n}\cup B_{3n} &= \{1,3,...,4n+1\}, \\ B_{1n}\cup B_{2n}\cup A_{2n} &= \{2,4,...,4n,4n+2\}. \end{aligned}$$

The sets $V_{in},\ i=1,2,3,$ form a partition of $V(C_{2n+1})$, the sets $E_{in},\ i=1,2,3,$ form a partition of $E(C_{2n+1})$ and

$$\begin{aligned} g_n(V(C_{2n+1}))\cup g_n^*(E(C_{2n+1})) &= A_{1n}\cup A_{2n}\cup A_{3n}\cup B_{1n}\cup B_{2n}\cup B_{3n} \\ &= \{1,2,...,4n+2\}. \end{aligned}$$

It is easy to verify that g_n is of *type* n. A *type* 1 labeling of C_7 and a *type* 3 labeling of C_9 are given in Figure 3. Thus we have proved the following lemma.

Lemma 2.3. *The cycle C_{2n+1} has type s labeling for each $s \in \mathcal{S}_{2n+1}$.*

Theorem 2.4. *The kC_{2n+1}-snake is a super mean graph for all $k \geq 1$ and $n \geq 3$.*

Proof. Let G be any kC_{2n+1}-snake. Then its string is of the form s_1, s_2, ..., s_{k-2}, where $s_i \in S_{2n+1}$ for each i. Let $B_1, B_2, B_3, ..., B_k$ be the consecutive blocks of G. First we label the blocks as given in Table 2. Then B_{i+1} and B_i are connected by identifying their vertices of common label $(4n+1)i$ +1. That is, the biggest label of B_i and the smallest label of B_{i+1}. Observe that the set of all vertex labels and the induced edge labels of the block B_i is $\{(4n+1)(i-1)+1, (4n+1)(i-1)+2, ..., (4n+1)(i-1)+(4n+2)\}$. It is evident that the set of all vertex labels and the induced edge labels of G is $\{1, 2, 3, ..., (4n+1)k+1\}$. Hence we have obtained a super mean labeling of G. A super mean labeling of a $6C_9$-snake is shown in Figure 4.

Blocks	Type used	offset x
B_1	any type	0
$B_{i+1}, 1 \leq i \leq k-2$	*type* s_i	$(4n+1)i$
B_k	any type	$(4n+1)k-(4n+1)$

Table 2:

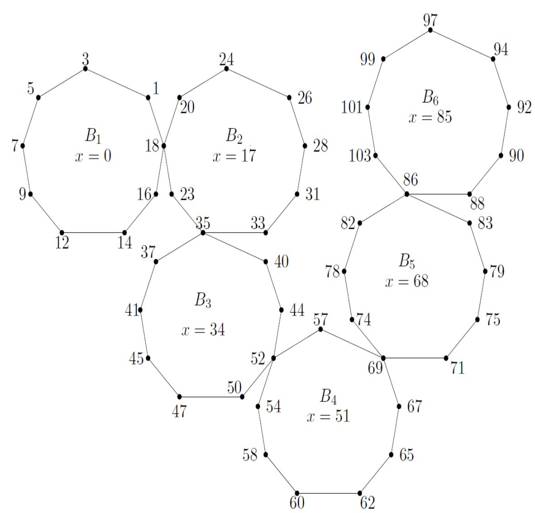


Figure 4: A super mean labeling of an $6C_9$-snake

□

Lemma 2.5. *If $n \geq 3$, the cycle C_{2n} has type s labeling for $2 \leq s \leq n$.*

Proof. Let C_{2n} be a cycle with consecutive vertices $v_0, v_1, v_2, ..., v_{2n-1}$. For $3 \leq n \leq 5$, the required labelings are shown in Figure 5.

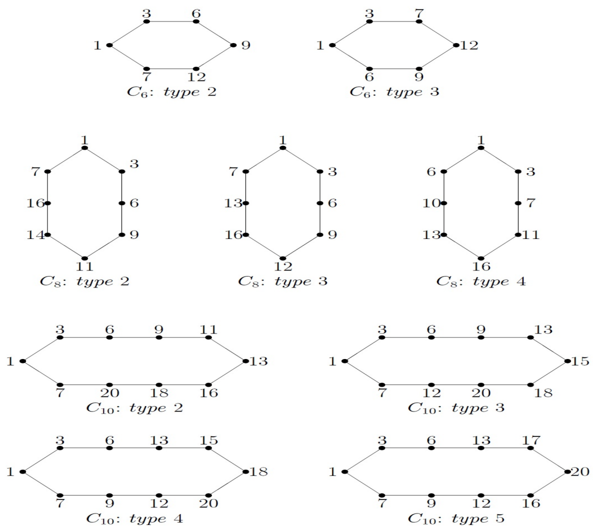


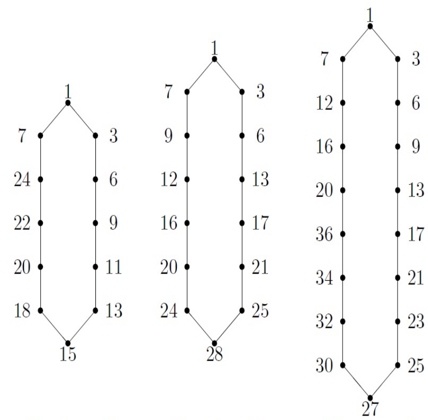
Figure 5: *Type s* labeling of C_{2n} for $3 \leq n \leq 5$

For $n \geq 6$, consider the labeling $g_s : V(C_{2n}) \longrightarrow \{1, 2, ..., 4n\}$ defined as follows:

$$g_2(v_j) = \begin{cases} 1 & : j = 0 \\ 7 & : j = 1 \\ 4n - 2j + 4 & : 2 \leq j \leq n - 1 \\ 4n - 2j + 3 & : n \leq j \leq 2n - 3 \\ 6 & : j = 2n - 2 \\ 3 & : j = 2n - 1 \end{cases},$$

for $3 \leq s \leq n - 2$

$$g_s(v_j) = \begin{cases} 1 & : j = 0 \\ 7 & : j = 1 \\ 4j + 4 & : 2 \leq j \leq s - 1 \\ 4n + 2s - 2j & : s \leq j \leq n - 1 \\ 4n + 2s - 2j - 1 & : n \leq j \leq 2n - s - 1 \\ 8n - 4j - 3 & : 2n - s \leq j \leq 2n - 3 \\ 6 & : j = 2n - 2 \\ 3 & : j = 2n - 1 \end{cases},$$



C_{12}: type 2 C_{14}: type 7 C_{18}: type 5

Figure 6: Some *type s* labeling of C_{2n} for $n \geq 6$

$$g_{n-1}(v_j) = \begin{cases} 1 & : j = 0 \\ 7 & : j = 1 \\ 9 & : j = 2 \\ 4j & : 3 \leq j \leq n-2 \\ 6n-2j-2 & : n-1 \leq j \leq n \\ 6n-2j-3 & : n+1 \leq j \leq n+2 \\ 8n-4j+1 & : n+3 \leq j \leq 2n-3 \\ 6 & : j = 2n-2 \\ 3 & : j = 2n-1 \end{cases}, \ \textit{and}$$

$$g_n(v_j) = \begin{cases} 1 & : j = 0 \\ 7 & : j = 1 \\ 9 & : j = 2 \\ 4j & : 3 \leq j \leq n \\ 8n-4j+1 & : n+1 \leq j \leq 2n-3 \\ 6 & : j = 2n-2 \\ 3 & : j = 2n-1 \end{cases}.$$

We show that g_s is a *type s* labeling of C_{2n}.

Case 1: $s = 2$

Let

$$V_{12} = \{v_0, v_1, v_{2n-2}, v_{2n-1}\},$$

$$V_{22} = \{v_j / 2 \leq j \leq n-1\},$$

$$V_{32} = \{v_j / n \le j \le 2n-3\},$$
$$E_{12} = \{v_{2n-1}v_0, v_0v_1, v_{2n-2}v_{2n-1}\},$$
$$E_{22} = \{v_1v_2\},$$
$$E_{32} = \{v_jv_{j+1} / 2 \le j \le n-1\},$$
$$E_{42} = \{v_jv_{j+1} / n \le j \le 2n-3\},$$
$$A_{k2} = g_2(V_{k2}),\ k = 1,2,3 \text{ and } B_{k2} = g_2^*(E_{k2}),\ k = 1,2,3,4.$$

Then

$$A_{12} = \{1,3,6,7\},$$
$$A_{22} = \{4n-2j+4 : 2 \le j \le n-1\}$$
$$= \{2n+6, 2n+8, ..., 4n-2, 4n\},$$
$$A_{32} = \{4n-2j+3 : n \le j \le 2n-3\}$$
$$= \{9, 11, ..., 2n+1, 2n+3\},$$
$$B_{12} = \{2,4,5\},$$
$$B_{22} = \{2n+4\},$$
$$B_{32} = \{4n-2j+3 : 2 \le j \le n-1\}$$
$$= \{2n+5, 2n+7, ..., 4n-3, 4n-1\},$$
$$B_{42} = \{4n-2j+2 : n \le j \le 2n-3\}$$
$$= \{8, 10..., 2n, 2n+2\},$$
$$A_{12} \cup B_{12} = \{1, 2, ..., 7\},$$
$$B_{42} \cup B_{22} \cup A_{22} = \{8, 10, ..., 4n-2, 4n\},$$
$$A_{32} \cup B_{32} = \{9, 11, ..., 4n-3, 4n-1\}.$$

The sets $V_{i2},\ i = 1,2,3$, form a partition of $V(C_{2n})$, the sets $E_{i2},\ i = 1,2,3,4$, form a partition of $E(C_{2n})$ and

$$g_2(V(C_{2n})) \cup g_2^*(E(C_{2n})) = \left(\bigcup_{i=1}^{3} A_{i2}\right) \cup \left(\bigcup_{i=1}^{4} B_{i2}\right)$$
$$= \{x+1, x+2, ..., x+4n\}.$$

It can be easily verified that g_2 is of *type* 2.
Case 2: $3 \le s \le n-2$
Let

$$V_{1s} = \{v_0, v_1, v_{2n-2}, v_{2n-1}\},$$
$$V_{2s} = \{v_j : \ 2 \le j \ \le s-1\},$$

$$V_{3s} = \{v_j : \ s \le j \ \le n-1\},$$
$$V_{4s} = \{v_j : \ n \le j \ \le 2n-s-1\},$$
$$V_{5s} = \{v_j : \ 2n-s \le j \ \le 2n-3\},$$
$$E_{1s} = \{v_{2n-3}v_{2n-2}, v_{2n-2}v_{2n-1}, v_{2n-1}v_0, v_0v_1\},$$
$$E_{2s} = \{v_jv_{j+1} : 1 \le j \ \le s-2 \},$$
$$E_{3s} = \{v_{s-1}v_s\},$$
$$E_{4s} = \{v_jv_{j+1} : \ s \le j \ \le n-1 \},$$
$$E_{5s} = \{v_jv_{j+1} : \ n \le j \ \le 2n-s-2 \},$$
$$E_{6s} = \{v_jv_{j+1} : \ 2n-s-1 \le j \ \le 2n-4 \},$$
$$A_{ks} = g_s(V_{ks}), \ 1 \le k \le 5 \ and \ B_{ks} = g_s^*(E_{ks}), 1 \le k \le 6.$$

Then

$$A_{1s} = \{1, 3, 6, 7\},$$
$$A_{2s} = \{4j+4 : \ 2 \le j \ \le s-1\}$$
$$= \{12, 16, ..., 4s-4, 4s\},$$
$$A_{3s} = \{4n+2s-2j : \ s \le j \ \le n-1\}$$
$$= \{2n+2s+2, 2n+2s+4, ..., 4n-2, 4n\},$$
$$A_{4s} = \{4n+2s-2j+1 : \ n \le j \ \le 2n-s-1\}$$
$$= \{4s+1, 4s+3, ..., 2n+2s-3, 2n+2s-1\},$$
$$V_{5s} = \{8n-4j-3 : \ 2n-s \le j \ \le 2n-3\}$$
$$= \{9, 13, ..., 4s-7, 4s-3\},$$
$$B_{1s} = \{2, 4, 5, 8\},$$
$$B_{2s} = \{4j+6 : \ 1 \le j \ \le s-2 \}$$
$$= \{10, 14, ...4s-6, 4s-2\},$$
$$B_{3s} = \{2n+2s\},$$
$$B_{4s} = \{4n+2s-2j-1 : \ s \le j \ \le n-1 \}$$
$$= \{2n+2s+1, 2n+2s+3, ..., 4n-3, 4n-1\},$$
$$B_{5s} = \{4n+2s-2j-2 : \ n \le j \ \le 2n-s-2 \}$$
$$= \{4s+2, 4s+4, ..., 2n+2s-4, 2n+2s-2\},$$
$$B_{6s} = \{8n-4j-1 : \ 2n-s-1 \le j \ \le 2n-4 \}$$
$$= \{11, 15, ..., 4s-5, 4s-1\},$$
$$A_{1s} \cup B_{1s} = \{1, 2, 3, 4, 5, 6, 7, 8\},$$

$$A_{5s} \cup B_{2s} \cup B_{6s} \cup A_{2s} = \{9, 10, ..., 4s\},$$

$$A_{4s} \cup B_{4s} \cup B_{5s} \cup B_{3s} \cup A_{3s} = \{4s+1, 4s+2, ..., 4n-1, 4n\}.$$

The sets $V_{is},\ i = 1, 2, 3, 4, 5$, form a partition of $V(C_{2n})$, the sets $E_{is},\ i = 1, 2, 3, 4, 5, 6$, form partition of $E(C_{2n})$ and
$g_s(V(C_{2n})) \cup g_s^*(E(C_{2n})) = \left(\bigcup_{i=1}^{5} A_{is}\right) \cup \left(\bigcup_{i=1}^{6} B_{is}\right) = \{1, 2, ..., 4n\}$. It is easy to verify that $g_s,\ 3 \le s \le n-1$ is of *type* s.
Case 3: $s = n-1$
Let

$$V_{1(n-1)} = \{v_0, v_1, v_2, v_{2n-2}, v_{2n-1}\},$$

$$V_{2(n-1)} = \{v_j :\ 3 \le j\ \ \le n-2\},$$

$$V_{3(n-1)} = \{v_{n-1}, v_n, v_{n+1}, v_{n+2}\},$$

$$V_{4(n-1)} = \{v_j :\ n+3 \le j\ \ \le 2n-3\},$$

$$E_{1(n-1)} = \{v_0v_1, v_0v_1, v_1v_2, v_2v_3, v_{2n-3}v_{2n-2}, v_{2n-2}v_{2n-1}\},$$

$$E_{2(n-1)} = \{v_jv_{j+1} :\ 3 \le j\ \ \le n-3\},$$

$$E_{3(n-1)} = \{v_jv_{j+1} :\ n-2 \le j\ \ \le n+1\},$$

$$E_{4(n-1)} = \{v_jv_{j+1} :\ n+2 \le j\ \ \le 2n-4\},$$

$A_{k(n-1)} = g_{(n-1)}(V_{k(n-1)}),\ k = 1, 2, 3, 4\ and\ B_{k(n-1)} = g^*_{(n-1)}(E_{k(n-1)}),\ k = 1, 2, 3, 4.$

Then

$$A_{1(n-1)} = \{1, 3, 6, 7, 9\},$$

$$A_{2(n-1)} = \{4j :\ 3 \le j\ \ \le n-2\},$$
$$= \{12, 16, ..., 4n-12, 4n-8\},$$

$$A_{3(n-1)} = \{4n-7, 4n-5, 4n-2, 4n\},$$

$$A_{4(n-1)} = \{8n-4j+1 :\ n+3 \le j\ \ \le 2n-3\}$$
$$= \{13, 17, ..., 4n-15, 4n-11\},$$

$$B_{1(n-1)} = \{2, 4, 5, 8, 10, 11\},$$

$$B_{2(n-1)} = \{4j+2 :\ 3 \le j\ \ \le n-3\ \}$$
$$= \{14, 18, .., 4n-14, 4n-10\},$$

$$B_{3(n-1)} = \{4n-6, 4n-4, 4n-3, 4n-1\},$$

$$B_{4(n-1))} = \{8n-4j-1 :\ n+2 \le j\ \ \le 2n-4\}$$
$$= \{15, 19, ..., 4n-13, 4n-9\},$$

$$A_{1(n-1)} \cup B_{1(n-1)} = \{1, 2, ..., 11\},$$

$$A_{2(n-1)} \cup A_{4(n-1)} \cup B_{2(n-1)} \cup B_{4(n-1)} = \{12, 13, ..., 4n-9, 4n-8\},$$

$$A_{3(n-1)} \cup B_{3(n-1)} = \{4n-7, 4n-6, ..., 4n-1, 4n\}.$$

The sets $V_{i(n-1)},\ i = 1,2,3,4$, form a partition of $V(C_{2n})$, the sets $E_{i(n-1)},\ i = 1,2,3,4$, form a partition of $E(C_{2n})$ and

$$g_{(n-1)}(V(C_{2n})) \cup g^*_{(n-1)}(E(C_{2n})) = \left(\bigcup_{i=1}^{4} A_{i(n-1)}\right) \cup \left(\bigcup_{i=1}^{4} B_{i(n-1)}\right)$$

$$= \{1, 2, ..., 4n\}.$$

It is easy to verify that g_{n-1} is of *type (n-1)*.

Case 4: $s = n$

Let

$$V_{1n} = \{v_0, v_1, v_2, v_{2n-2}, v_{2n-1}\},$$

$$V_{2n} = \{v_j :\ 3 \le j\ \le n\},$$

$$V_{3n} = \{v_j :\ n+1 \le j\ \le 2n-3\},$$

$$E_{1n} = \{v_0v_1, v_0v_1, v_1v_2, v_2v_3, v_{2n-3}v_{2n-2}, v_{2n-2}v_{2n-1}, v_{2n-1}v_0\},$$

$$E_{2n} = \{v_jv_{j+1} :\ 3 \le j\ \le n-1\},$$

$$E_{3n} = \{v_jv_{j+1} :\ n \le j\ \le 2n-4\},$$

$$A_{kn} = g_n(V_{kn}),\ k = 1,2,3 \ \textit{and}\ B_{kn} = g^*_n(E_{kn}),\ k = 1,2,3.$$

Then

$$A_{1n} = \{1, 3, 6, 7, 9\},$$

$$A_{2n} = \{4j :\ 3 \le j\ \le n\}$$
$$= \{12, 16, ..., 4n-4, 4n\},$$

$$A_{3n} = \{8n-4j+1 :\ n+1 \le j\ \le 2n-3\}$$
$$= \{13, 17, ..., 4n-7, 4n-3\},$$

$$B_{1n} = \{2, 4, 5, 8, 10, 11\},$$

$$B_{2n} = \{4j+2 :\ 3 \le j\ \le n-1\ \}$$
$$= \{14, 18, .., 4n-6, 4n-2\},$$

$$B_{3n} = \{8n-4j-1 :\ n \le j\ \le 2n-4\}$$
$$= \{15, 19, ..., 4n-5, 4n-1\},$$

$$A_{1n} \cup B_{1n} = \{1, 2, ..., 11\},$$

$$A_{2n} \cup A_{3n} \cup B_{2n} \cup B_{3n} = \{12, 13, ..., 4n-9, 4n-8\}.$$

The sets $V_{in},\ i = 1, 2, 3$, form a partition of $V(C_{2n})$, the sets $E_{in}, i = 1, 2, 3$, form a partition of $E(C_{2n})$ and

$$g_n(V(C_{2n})) \cup g_n^*(E(C_{2n})) = \left(\bigcup_{i=1}^{3} A_{in}\right) \cup \left(\bigcup_{i=1}^{3} B_{in}\right) = \{1, 2, ..., 4n\}.$$

It is easy to verify that g_n is of *type* n. Some *type* s labelings of C_{2n} are illustrated in Figure 6. Hence the theorem. □

In [5], it was proved that C_4 is not a super mean graph and hence it has no *type* s labeling for any s.

Theorem 2.6. *The kC_{2n}-snake with string $s_1, s_2, s_3, ...s_{k-2}$ where $s_i \geq 2$ for each i is a super mean graph for all $k \geq 1$ and $n \geq 3$.*

Proof. Let G be any kC_{2n}-snake with string $s_1, s_2, s_3, ...s_{k-2}$ where $s_i \geq 2$ for each i. Let $B_1, B_2, B_3, ..., B_k$ be the consecutive blocks of G. Using Lemma 2.5, first we label the blocks as given in Table 3.

Blocks	Type used	offset x
B_1	any type	0
$B_{i+1}, 1 \leq i \leq k-2$	*type* s_i	$(4n-1)i$
B_k	any type	$(4n-1)k - (4n-1)$

Table 3:

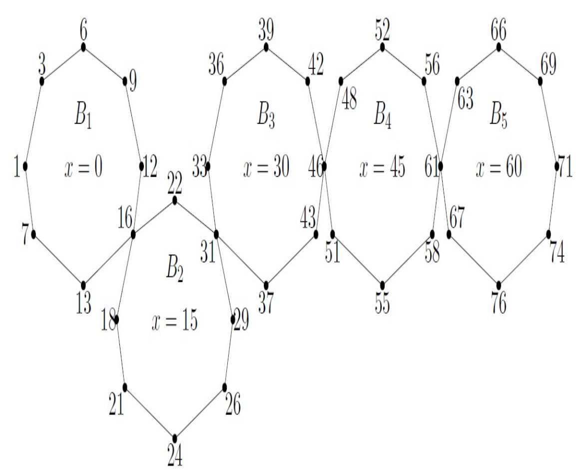


Figure 7: A super mean labeling of a $5C_8$-snake

Then B_{i+1} and B_i are connected by identifying their vertices of common label $(4n-1)i + 1$. Observe that the set of all vertex labels and the induced edge labels

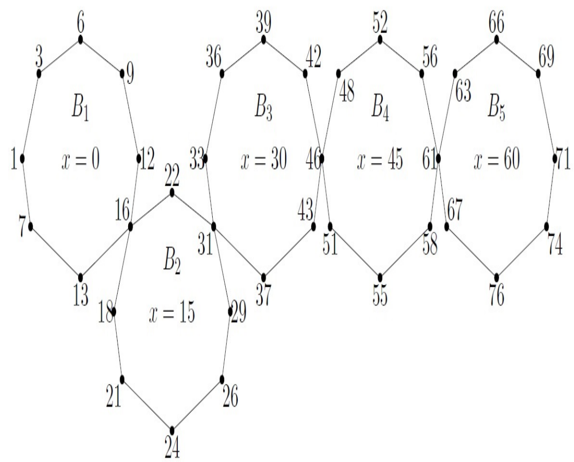


Figure 8: A super mean labeling of a $4C_{14}$-snake

of the block B_i is
$\{(4n-1)(i-1)+1, (4n-1)(i-1)+2, ..., (4n-1)(i-1)+4n\}$. It is evident that the set of all vertex labels and the induced edge labels of G is $\{1, 2, 3, ..., (4n-1)k+1\}$. Hence we have obtained a super mean labeling of G. Super mean labeling of a $5C_8$-snake and a $4C_{14}$-snake are shown in Figure 7 and Figure 8 respectively. □

Notation 2.7. *A kC-snake is defined as a connected graph in which each of the k many block is isomorphic to a cycle C_n for some n and the block-cut point graph is a path. It is denoted by $CS(n_1, n_2, n_3, ..., n_k)$ where $B_1, B_2, B_3, ..., B_k$ are the consecutive blocks and B_i is isomorphic to C_{n_i} . By applying the same methods used to obtain the string of a kC_n-snake , we can show that any kC -snake can be represented by a string of integers, $s_1, s_2, s_3, ..., s_{k-2}$ where $s_i \in \mathcal{S}_{n_{i+1}}$.*

Theorem 2.8. *The kC-snake $CS(n_1, n_2, n_3, ..., n_k)$ with the string $s_1, s_2, s_3, ..., s_{k-2}$ where $n_i \geq 3$, $n_i \neq 4$, and $s_i \geq 2$ if $n_{i+1} \equiv 0 (mod\ 2)$ is a super mean graph.*

Proof.

Blocks	Type used	offset x
B_1	any type	0
$B_{i+1}, 1 \leq i \leq k-2$	*type* s_i	$\sum_{r=1}^{i}(2n_r - 1)$
B_k	any type	$\sum_{r=1}^{k-1}(2n_r - 1)$

Table 4:

Let G be any kC-snake. Then its string is of the form $s_1, s_2, s_3, ...s_{k-2}$, where $s_i \in \left\{1, 2, ..., \left\lfloor \frac{n_{i+1}}{2} \right\rfloor\right\}$. Let $B_1, B_2, B_3, ..., B_k$ be the consecutive blocks of G. Using Lemma 2.3 and Lemma 2.5, first we label the blocks as given in Table 4. Then B_{i+1}

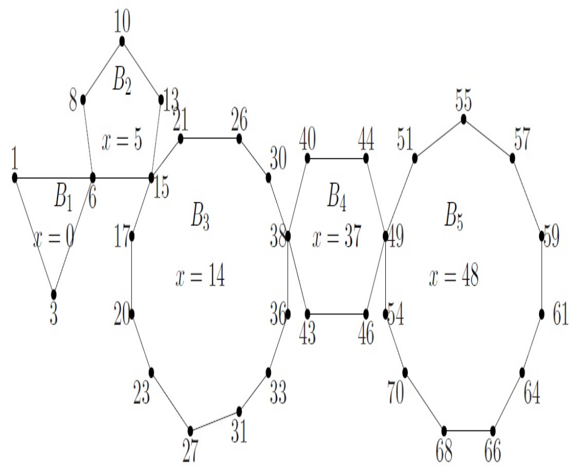


Figure 9: A super mean labeling of a $CS(3, 5, 12, 6, 11)$

and B_i are connected by identifying their vertices of common label $\sum_1^i (2n_r - 1) + 1$. Observe that the set of all vertex labels and the induced edge labels of the block B_i is $\left\{\sum_1^{i-1}(2n_r - 1) + 1, \sum_1^{i-1}(2n_r - 1) + 2, ..., \sum_1^{i-1}(2n_r - 1) + 2n_i\right\}$.
It is easy to verify that the set of all vertex labels and the induced edge labels of G is $\left\{1, 2, 3, ..., \sum_1^k (2n_r - 1) + 1\right\}$. Thus we have obtained a super mean labeling of G. A super mean labeling of a $CS(3, 5, 12, 6, 11)$ with string $1, 4, 3$ is shown in Figure 9.

□

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