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Research Article

On Extension of Lauricella Function with Properties

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Abstract: In this paper we first establish the two new functions $S_t(\lambda_{1,}...,\lambda_2, p, a, b_{1,}...,b_n)$ and $S_t(\lambda_{1,}... ,\lambda_n, q, a, b_{1,}... ,b_n)$ using generalized Lauricella function [7] then we discuss the fractional integral and differential properties, integral representations and Laplace transform of these functions. Further we have mentioned the particular cases of our results which are new & interesting by themselves.

Keywords: Lauricella function, Fractional integration& differentiation, Laplace transforms, Recurrence relations.

1. Introduction

In recent years, some generalization of the well known special functions have been considered by several authors [2], [8], [12], [9], [10], [11].

The Gauss hyper geometric function is defined [4] as

$$ {}_2F_1(a,b,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k}\frac{z^k}{k!} \qquad \{|z|<1, c\neq 0,-1,-2...\} \qquad \dots (1)$$

And generalized hypergeometric function is defined as [1]

$$ {}_pF_q(a_1...a_p;b_1...b_q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k}\frac{z^k}{k!}, \quad \left[p=q+1,\ |z|<1\right] \qquad \dots (2)$$

and no denominator parameter equal to zero or negative integer.

The fruitful nature of the theory of single hypergeometric functions led to generalisation involving double series. Appell (1880) was the first author to treat this matter on a systemic basis, and he defined the four functions F_1, F_2, F_3, F_4 in [4] in which F_1 is given as

$$F_1(a;b_1,b_2;c;x;y)=\sum_{n_1,n_2=0}^{\infty}\frac{(a)_{n_1+n_2}(b_1)_{n_1}(b_2)_{n_2}}{(c)_{n_1+n_2}}\frac{x^{n_1}}{n_1!}.\frac{y^{n_2}}{n_2!}\ \text{where}\left[|x|<1,|y|<1\right] \quad \ldots (3)$$

Further Lauricella proceeded to define and study the four important functions in multiple series representation as[4] in which $F_D^{(n)}$ function is defined as [4].

$$F_D^{(n)}(a;b_1...b_n;c;x_1...x_n)=\sum_{k_1...k_n=0}^{\infty}\frac{(a)_{k_1+...+k_n}(b_1)_{k_1}...(b_n)_{k_n}}{(c)_{k_1+...+k_n}}\frac{(x_1)^{k_1}}{k_1!}...\frac{(x_n)^{k_n}}{k_n!} \quad \ldots (4)$$

where $|x_1|<1,|x_2|<1,\ |x_3|<1,...,|x_n|<1$

Further M. Garg & R. Mishra generalized the τ-generalization of Lauricella function of $F_D^{(n)}$ as [7].

$$F_D^{(n)(\tau_1...\tau_n)}(a;b_1,...;b_n;c;x_1,...;x_n)=\frac{\Gamma c}{\Gamma a}\sum_{k_1...k_n=0}^{\infty}\frac{\Gamma(a+\tau_1k_1+...+\tau_nk_n)\ (b_1)_{k_1}...(b_n)_{k_n}}{\Gamma(c+\tau_1k_1+...+\tau_nk_n)}\frac{(x_1)^{k_1}}{k_1!}...\frac{(x_n)^{k_n}}{k_n!} \quad \ldots (5)$$

where $|x_1|<1...|x_n|<1$ and $\tau_i>0$

The Riemann liouville fractional integral of order p is defined as [5] for R(p) > 0.

$$I^pf(t)=\frac{1}{\Gamma p}\int_0^t(t-\varepsilon)^{p-1}f(\varepsilon)d\varepsilon \quad \ldots (6)$$

And the fraction differential operator of order q is defined as [5].

$$D^qf(t)=D^n\left[I^{n-q}f(t)\right] \quad \ldots (7)$$

Where R(q) > 0 and n is the smallest integer with the probability that n > R(q). The Laplace transform of function f(t) is defined as [6].

$$L(f(t))=\int_0^{\infty}e^{-st}f(t)dt \quad \ldots (8)$$

2. Fractional Operators on Reneralized Lauricella Function $F_D^{(n)(\tau_1,...,\tau_n)}(a;b_1,...,b_n;c;x_1,...,x_n)$

Consider a function

$$f(t)=\frac{1}{\Gamma a}\sum_{k_1,...,k_n=0}^{\infty}\frac{\Gamma(a+k_1+...+k_n)\ (b_1)_{k_1}...(b_n)_{k_n}}{(k_1+k_2+...+k_n)!}.\frac{(\lambda_1t)^{k_1}}{k_1!}...\frac{(\lambda_nt)^{k_n}}{k_n!} \quad \ldots (9)$$

where a, $b_1, b_2, \ldots, b_n \in c$, $R(a) > 0$, $R(b_1), \ldots, R(b_n) > 0$ and $\lambda_1 \ldots \lambda_n$ are arbitrary constant s.t. $|\lambda_1 t| < 1 \ldots |\lambda_n t| < 1$

On applying (9) in (6) we get

$$I^p f(t) = \frac{1}{\Gamma p}\int_0^t (t-\varepsilon)^{p-1} \cdot \frac{1}{\Gamma a}\sum_{k_1 \ldots k_n=0}^{\infty} \frac{\Gamma(a+k_1+\ldots+k_n)(b_1)_{k_1}\ldots(b_n)_{k_n}}{(k_1+\ldots+k_n)!}\frac{(\lambda_1\varepsilon)^{k_1}}{k_1!}\cdots\frac{(\lambda_n\varepsilon)^{k_n}}{k_n!}d\varepsilon$$

After simplification

$$= \frac{t^p}{\Gamma(p+1)}\left\{\frac{\Gamma(p+1)}{\Gamma a}\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{\Gamma(a+k_1+\ldots+k_n)}{\Gamma(p+1+k_1+\ldots+k_n)} \times (b_1)_{k_1}\ldots(b_n)_{k_n} \cdot \frac{(\lambda_1 t)^{k_1}}{k_1!}\cdots\frac{(\lambda_n t)^{k_n}}{k_n!}\right.$$

$$= \frac{t^p}{\Gamma(p+1)} F_D^{(n)(1\ldots1)}(a, b_1, \ldots, b_n; p+1; \lambda_1 t, \ldots, \lambda_n t)$$

$$= \frac{t^p}{\Gamma(p+1)} F_D^{(n)}(a, b_1, \ldots, b_n, p+1; \lambda_1 t, \ldots, \lambda_n t) = S_t(\lambda_1, \ldots, \lambda_n, p, a, b_1, \ldots, b_n) \quad \ldots\ldots (10)$$

Further applying (9) in (7), we have

$$D^q f(t) = D^n\left[\frac{t^{n-q}}{\Gamma a}\sum_{k_1\ldots k_n=0}^{\infty}\frac{\Gamma(a+k_1+\ldots+k_n)\ (b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma(n-q+1+k_1+\ldots+k_n)}\frac{(\lambda_1 t)^{k_1}}{k_1!}\cdots\frac{(\lambda_n t)^{k_n}}{k_n!}\right.$$

Which yields

$$D^q f(t) = \frac{t^{-q}}{\Gamma(1-q)} F_D^{(n)(1\ldots1)}(a, b_1, \ldots, b_n;\ 1-q;\ \lambda_1 t, \ldots, \lambda_n t)$$

$$= \frac{t^{-q}}{\Gamma(1-q)} F_D^{(n)}(a, b_1, \ldots, b_n; 1-q; \lambda_1 t, \ldots, \lambda_n t) = S_t(\lambda_1 \ldots \lambda_n, -q, a, b_1 \ldots b_n) \quad \ldots(11)$$

3. Some Integral Representations of Generalized Lauricella Function $F_D^{(n)(\tau_1,\ldots,\tau_n)}(a; b_1, \ldots, b_n; c, x_1, \ldots, x_n)$

Theorem 3.1:

For $\tau_i > 0$, $R(a) > 0$, $R(b_i) > 0$, $R(m) > 0$and $R(m-a-b_i) > 0$ *where* $i = 1, 2 \ldots n$

$$\int_0^1 t^m F_D^{(n)(\tau_1,\ldots,\tau_n)}(a, b_1, \ldots, b_n; m; \lambda_1 t^{\tau_1}, \ldots, \lambda_n t^{\tau_n})dt$$

$$=\frac{1}{m}F_D^{(n)(\tau_1,\dots,\tau_n)}\left(a,b_1\dots b_n;m+1;\lambda_1\dots\lambda_n\right)$$

$$-\frac{1}{m(m+1)}F_D^{(n)(\tau_1,\dots,\tau_n)}\left(a,b_1,\dots,b_n;m+2;\ \lambda_1 t^{\tau_1},\dots,\lambda_n t^{\tau_n}\right)$$

Proof: To prove the above theorem, representing $F_D^{(n)(\tau_1,\dots,\tau_n)}\left(a;b_1,\dots,b_n;c,x_1,\dots,x_n\right)$ using equation (5), applying the integration with respect to t between the limits 0 to 1 , and using $\frac{1}{x+1}=\frac{1}{x}-\frac{1}{x(x+1)}$,we early arrived the required result.

Theorem 3.2: *If a, c, b_i, $\tau_i \in$ Cs.t. R(a) > 0, R(b_i) > 0 R(c) > 0 and R(c-τ_i) > 0, μ> 0 and* $|x|<1$ *then*

$$F_D^{(n)(\tau_1,\dots,\tau_n)}(a,b_1,\dots,b_n;c\ ;\lambda_1 x,\dots,\lambda_n x)$$

$$=\mu x^{\tau_1+\dots+\tau_n-c}\int_0^\infty \exp.\left(-\frac{t^\mu}{x^\mu}\right)t^{c-(\tau_1+\dots+\tau_n)-1}\ .$$

$$\sum_{k_1\dots k_n=0}^{\infty}\frac{\Gamma c}{\Gamma a}\frac{\Gamma\left(a+\tau_1 k_1+\dots+\tau_n k_n\right)}{\Gamma\left(c+\tau_1 k_1+\dots+\tau_n k_n\right)}\left(b_1\right)_{k_1}\dots\left(b_n\right)_{k_n}\ .$$

$$\frac{1}{\Gamma\left(\frac{c-(\tau_1+\dots+\tau_n)+(k_1+\dots+k_n)}{\mu}\right)}\frac{\left(\lambda_1 t\right)^{k_1}}{k_1!}\dots\frac{\left(\lambda_n t\right)^{k_n}}{k_n!}dt$$

Proof: On taking right hand side of the above theorem (3.2), putting $\left(\frac{t}{x}\right)^\mu$ as new parameter and using the gamma function property as $\int_0^\infty e^{-x}x^{n-1}dx=\Gamma n$, we get required result of(3.2).

Theorem 3.3: *If a, b_i, c, $\tau \in C$ s.t.R(a) > 0, R(b_i) > 0, R(c) > 0 and R(c-τ) > 0 and* $|x|<1$ *then*

$$F_D^{n)(\tau,\dots,\tau)}\left(a,b_1,\dots,b_n;c;x_1,\dots,x_n\right)=$$

$$\frac{\Gamma c}{\tau\Gamma\tau\Gamma(c-\tau)}\int_0^1\left(1-t^{1/\tau}\right)^{c-\tau-1}F_D^{(n)(\tau,\dots,\tau)}\left(a,b_1,\dots,b_n;\ \tau;\ tx_1,\dots,tx_n\right)dt$$

Proof: R.H.S.= $\frac{\Gamma c}{\tau\Gamma\tau\Gamma\left(c-\tau\right)}\sum_{k_1\dots k_n=0}^{\infty}\frac{\Gamma\tau}{\Gamma a}\frac{\Gamma\{a+\tau(k_1+\dots+k_n)\}}{\Gamma\{\tau+\tau(k_1+\dots+k_n)\}}\left(b_1\right)_{k_1}\dots\left(b_n\right)_{k_n}\frac{\left(x_1\right)^{k_1}}{k_1!}\dots\frac{\left(x_n\right)^{k_n}}{k_n!}$.

$$\tau\int_0^1(1-\mu)^{c-\tau-1}\mu^{\tau+(k_1+\dots+k_n)\tau-1}d\mu$$

Let $t^{1/\tau} = \mu$ and using the result $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

We get, = $F_D^{(n)(\tau,...,\tau)}\left(a, b_1,...,b_n; c; x_1,...,x_n\right)$

Theorem 3.4: *If a, b_i, c, τ $\in$ C s.t. R(a) > 0, R(c) > 0, R(b_i) > 0 & R(c- τ) > 0 and* $|x_i| < 1$ *where i = 1 to n*

Then $F_D^{(n)(\tau,...,\tau)}(a, b_1,...,b_n, c, x_1,...,x_n)$ =

$$\frac{\Gamma c}{\Gamma\tau\Gamma(c-\tau)}\int_0^1 t^{\tau-1}(1-t)^{c-\tau-1} F_D^{(n)(\tau,...,\tau)}\left(a, b_1,...,b_n; c-\tau; x_1(1-t)^{\tau},...,x_n(1-t)^{\tau}\right)$$

Proof: On taking R.H.S., we obtain

$$\frac{\Gamma c}{\Gamma\tau\Gamma(c-\tau)}\int_0^1 t^{\tau-1}(1-t)^{c-\tau-1}\left(\frac{\Gamma(c-\tau)}{\Gamma a}\sum_{k_1...k_n=0}^{\infty}\frac{\Gamma\{a+\tau(k_1+...+k_n)\}}{\Gamma\{(c-\tau)+\tau(k_1+...+k_n)\}}(b_1)_{k_1}...(b_n)_{k_n}\times\frac{\left(x_1(1-t)^{\tau}\right)^{k_1}}{k_1!}...\frac{\left(x_n(1-t)^{\tau}\right)^{k_n}}{k_n!}\right)dt$$

$$= \frac{\Gamma c}{\Gamma\tau}.\frac{1}{\Gamma a}\sum_{k_1...k_n=0}^{\infty}\frac{\Gamma\{a+\tau(k_1+...+k_n)\}}{\Gamma\{(c-\tau)+\tau(k_1+...+k_n)\}}(b_1)_{k_1}...(b_n)_{k_n}\frac{(x_1)^{k_1}}{k_1!}...\frac{(x_n)^{k_n}}{k_n!}\ .$$

$$\int_0^1 t^{\tau-1}(1-t)^{c-\tau+\tau(k_1+...+k_n)-1}dt$$

Further using beta function property, we get

$$= F_D^{(n)(\tau,...,\tau)}\left(a, b_1,...,b_n; c; x_1,...,x_n\right)$$

4. Properties of Function $S_t(\lambda_1,\ldots,\lambda_n, p; a, b_1,\ldots,b_n)$ and $S_t(\lambda_1,\ldots,\lambda_n, -q, a, b_1,\ldots,b_n)$

Theorem 4.1: *If a, $b_1,\ldots,b_n \in$ c, R(a)>0, R(b_1)>0,…,R(b_n)>0, and $\lambda_1,\ldots,\lambda_n$ is arbitrary constant s.t* $|\lambda_1 t| < 1 ... |\lambda_n t| < 1$

$$I^{\nu} S_t(\lambda_1,\ldots,\lambda_n, p, a, b_1,\ldots,b_n) = S_t(\lambda_1,\ldots,\lambda_n, p+\nu, a, b_1,\ldots,b_n) \qquad \ldots(12)$$

$$D^{\nu} S_t(\lambda_1,\ldots,\lambda_n, p, a, b_1,\ldots,b_n) = S_t(\lambda_1,\ldots,\lambda_n, p-\nu, a, b_1,\ldots,b_n) \qquad \ldots(13)$$

for $n \in N$ and $\alpha_1\ldots\alpha_n$ are any constant, then

$L[S_t(\lambda_1,\ldots,\lambda_n, p, -n, \alpha_1+n-1,\ldots,\alpha_n+n-1)]=$

$$\frac{1}{s^{p+1}} F_D^{(n)(1\ldots1)}\left[-n,(\alpha_1+n-1),\ldots,(\alpha_n+n-1);-;\frac{\lambda_1}{s},\ldots,\frac{\lambda_n}{s}\right] \quad \ldots (14)$$

Proof: To prove the result (12), we take left hand side as

$$I^{\nu} S_t(\lambda_1,\ldots,\lambda_n,\ p,\ a,\ b_1,\ldots,\ b_n) = \frac{1}{\Gamma\nu}\int_0^t (t-\varepsilon)^{\nu-1} s_{\varepsilon}(\lambda_1,\ldots,\lambda_n,p,a,b_1,\ldots,b_n)d\varepsilon$$

$$= \frac{1}{\Gamma\nu}\int_0^t (t-\varepsilon)^{\nu-1}\left[\frac{\varepsilon^p}{\Gamma a}\sum_{k_1\ldots k_n=0}^{\infty}\frac{\Gamma(a+k_1+\ldots+k_n)\ (b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma(p+1+k_1+\ldots+k_n)}\cdot\frac{(\lambda_1\varepsilon)^{k_1}}{k_1!}\cdots\frac{(\lambda_n\varepsilon)^{k_n}}{k_n!}\right]d\varepsilon$$

$$\frac{1}{\Gamma\nu}\cdot\frac{1}{\Gamma a}\sum_{k_1\ldots k_n=0}^{\infty}\frac{\Gamma(a+k_1+\ldots+k_n)(b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma(p+1+k_1+\ldots+k_n)}\cdot\frac{(\lambda_1)^{k_1}}{k_1!}\cdots\frac{(\lambda_n)^{k_n}}{k_n!}$$

$$\cdot\int_0^t (t-\varepsilon)^{\nu-1}\varepsilon^{p+k_1+\ldots+k_n}d\varepsilon$$

On substituting ε=ts and using beta function, we get

$$\frac{t^{p+\nu}}{\Gamma(p+\nu+1)}\cdot\left[\frac{\Gamma(p+\nu+1)}{\Gamma a}\sum_{k_1\ldots k_n=0}^{\infty}\frac{\Gamma(a+k_1+\ldots+k_n)\ (b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma(p+\nu+1+k_1+\ldots+k_n)}\frac{(\lambda_1 t)^{k_1}}{k_1!}\cdots\frac{(\lambda_n t)^{k_n}}{k_n!}\right]$$

$$= S_t(\lambda_1,\ldots,\lambda_n,\ p+\nu,\ a,\ b_1,\ldots,b_n)$$

Further, using fractional differential operator from equation (7) in left hand side of the result (13), we get

$$D^{\nu} S_t(\lambda_1,\ldots,\lambda_n,\ p,a,b_1,\ldots b_n) = D^n\{\ I^{n-\nu}\ S_t(\lambda_1,\ldots,\lambda_n,\ p,\ a,\ b_1,\ldots,b_n)]$$

$$= D^n\left[\frac{t^{n-\nu+p}}{\Gamma(n-\nu+p+1)}F_D^{(n)(1,\ldots,1)}(a,b_1,\ldots,b_n,n-\nu+p+1,\lambda_1 t,\ldots,\lambda_n t)\right]$$

On differentiating the above result n times, we arrived at the required result (13).
Again setting $n\in N$, $a=-n$, $b_i=\alpha_i+n-1$where α_i is any constant and $i = 1\ldots n$in

$S_t(\lambda_1,\ldots,\ \lambda_n,\ p,\ a,\ b_1,\ldots,b_n)$ using (8) in (14) ,we get

$$\int_0^{\infty} e^{-st}\left\{\frac{t^p}{\lceil p+1}F_D^{(n)(1,\ldots,1)}(-n,\alpha_1+n-1,\ldots,\alpha_n+n-1,p+1,\lambda_1 t,\ldots,\lambda_n t)\right\}dt$$

$$= \frac{1}{s^{p+1}}F_D^{(n)(1,\ldots,1)}\left[-n,(\alpha_1+n-1),\ldots,(\alpha_n+n-1);-;\frac{\lambda_1}{s},\ldots,\frac{\lambda_n}{s}\right]$$

which is Lauricella function and it gives the proof of (14).

Theorem 4.2: Let a, $b_1,\ldots,b_n \in$ c, R(a)>0, $R(b_1)>0,\ldots,R(b_n)>0$and λ_i constant s.t. $|\lambda_i t| < 1$ where i=1…n, R(q)<1 then

$$I^{\mu}S_t(\lambda_1,\ldots,\lambda_n, -q, a, b_1,\ldots,b_n) = S_t(\lambda_1,\ldots,\lambda_n, q-\mu, a, b_1,\ldots,b_n) \qquad \ldots (15)$$

$$D^{\mu}S_t(\lambda_1,\ldots,\lambda_n, -q, a, b_1,\ldots,b_n) = S_t(\lambda_1,\ldots,\lambda_n, -\mu-q, a, b_1,\ldots,b_n) \qquad \ldots (16)$$

For $n \in N$, $\alpha_1,\ldots,\alpha_n$ are any constants

$$L\,[S_t(\lambda_1,\ldots,\lambda_n, -q, -n, \alpha_1+n-1,\ldots,\alpha_n+n-1)] =$$

$$\frac{1}{s^{1-q}} F_D^{(n)(1,\ldots,1)}\left(-n, \alpha_1+n-1,\ldots,\alpha_n+n-1; -; \frac{\lambda_1}{s},\ldots,\frac{\lambda_n}{s}\right) \qquad \ldots (17)$$

Proof: $I^{\mu}s_t\left(\lambda_1\ldots\lambda_n, -q, a, b_1\ldots b_n\right) = \frac{1}{\Gamma\mu}\int_0^t (t-\varepsilon)^{\mu-1} s_{\varepsilon}\left(\lambda_1\ldots\lambda_n, -q, a, b_1\ldots b_n\right) d\varepsilon$

$$= \frac{1}{\Gamma\mu}\int_0^t (t-\varepsilon)^{\mu-1}\left\{\frac{\varepsilon^{-q}}{\Gamma a}\sum_{k_1\ldots k_n=0}^{\infty}\frac{\Gamma(a+k_1+\ldots+k_n)\ (b_1)_{k_1}\ldots(b_n)_{k_n}}{\Gamma(1-q+k_1+\ldots+k_n)}\frac{(\lambda_1\varepsilon)^{k_1}}{k_1!}\cdots\frac{(\lambda_n\varepsilon)^{k_n}}{k_n!}\right. d\varepsilon$$

Replacing ε=ts and using beta function, we obtain

$$= \frac{t^{\mu-q}}{\Gamma(\mu-q+1)} F_D^{(n)(1\ldots1)}\left(a, b_1,\ldots,b_n, \mu-q+1, \lambda_1 t,\ldots,\lambda_n t\right) = S_t(\lambda_1,\ldots, \lambda_n, \mu-q, a, b_1,\ldots,b_n)$$

Further, using fractional differential equation from (7) in left hand side of the result (15), we get

$$D^{\mu} S_t(\lambda_1,\ldots,\lambda_n, -q, a, b_1,\ldots,b_n) = D^n[\, I^{n-\mu}S_t\left(\lambda_1,\ldots,\lambda_n, -q, a, b_1,\ldots,b_n\right)$$

On differentiating n times we arrived the required result (16)

$$=S_t(\lambda_1,\ldots,\lambda_n, -\mu-q, a, b_1,\ldots,b_n)$$

Again setting $n \in N$, $\alpha_1,\ldots\ldots,\alpha_n$ are constant. Replacing a=-n, $b_i = \alpha_i + n - 1$ where i=1…n Using (8) in (17)

$$= \int_0^{\infty} e^{-st}\frac{t^{-q}}{\Gamma(1-q)} F_D^{(n)(1,\ldots,1)}(-n, \alpha_1+n-1,\ldots, \alpha_n+n-1; 1-q, \lambda_1 t,\ldots,\lambda_n t)dt$$

After simplification we get

$$= \frac{1}{s^{1-q}} F_D^{(n)(1,\dots,1)}\left[-n,\alpha_1+n-1,\dots,\alpha_n+n-1;-;\frac{\lambda_1}{s},\dots,\frac{\lambda_n}{s}\right]$$

5. Special Cases

(i) On taking $n=2$ and $c=m+1$ in generalized Lauricella function [7], we get the τ-generalizations of appell function and further suitably specializing the parameters, we obtain some recurrence relations of appell functions.

Theorem 5.1: If $R(\tau_1),R(\tau_2)>0,R(a)>0,R(b_1),R(b_2)>0,R(m)>0,|x|<1,|y|<1$ then

$$(m+1)F_1^{\tau_1,\tau_2}(a,b_1,b_2;m+1;x,y)-F_1^{\tau_1,\tau_2}(a,b_1,b_2;m+2;x,y)$$

$$=\frac{\tau_1^2}{(m+2)}x^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x^2}+\frac{\tau_2^2}{(m+2)}y^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial y^2}+\frac{2\tau_1\tau_2}{(m+2)}xy\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x\partial y}$$

$$+\frac{\tau_1}{(m+2)}[\tau_1+2(m+1)]x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}+\frac{\tau_2}{(m+2)}[\tau_2+2(m+1)]y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}+mF_1^{\tau_1,\tau_2}$$

Proof: Consider

$$F_1^{\tau_1,\tau_2}(a,b_1,b_2,m+2,x,y) = \frac{\Gamma(m+2)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)\ (b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+2+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!}$$

$$=\frac{\Gamma(m+2)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\left\{\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)\ (b_1)_{k_1}(b_2)_{k_2}}{[(m+1)+\tau_1k_1+\tau_2k_2][m+\tau_1k_1+\tau_2k_2]\Gamma(m+\tau_1k_1+\tau_2k_2)}\right\}$$

On rewriting the left side of the above equation by using the property $\Gamma(x+1)=x\Gamma x$, we have

$$\frac{\Gamma(m+2)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\left\{\frac{1}{m+\tau_1k_1+\tau_2k_2}-\frac{1}{(m+1)+\tau_1k_1+\tau_2k_2}\right\}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!}$$

$$F_1^{\tau_1,\tau_2}(a,b_1,b_2;m+2;x,y)=(m+1)F_1^{(\tau_1,\tau_2)}(a,b_1,b_2;m+1;x,y)$$

$$-\frac{\Gamma(m+2)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{[(m+1)+\tau_1k_1+\tau_2k_2]\Gamma(m+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!} \quad \dots (18)$$

For the convenience, we denote the II term of (18) with U.

$$U=\frac{\Gamma(m+2)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)}{(m+1+\tau_1k_1+\tau_2k_2)\Gamma(m+\tau_1k_1+\tau_2k_2)}(b_1)_{k_1}(b_2)_{k_2}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!}$$

$$= \left(m+1\right)F_1^{\tau_1,\tau_2}\left(a,b_1,b_2;m+1;x,y\right) - F_1^{\tau_1,\tau_2}\left(a,b_1,b_2;m+2;x,y\right) \qquad \ldots (19)$$

Taking L.H.S. of (19) and using $\frac{1}{s} = \frac{1}{s(s+1)} + \frac{1}{(s+1)}$ where $s=(m+1)+\tau_1 k_1+\tau_2 k_2$, we have

$$U = \frac{\Gamma(m+2)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\frac{(m+\tau_1k_1+\tau_2k_2)\Gamma(a+\tau_1k_1+\tau_2k_2)}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}(b_1)_{k_1}(b_2)_{k_2}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!}$$

$$+\frac{\Gamma(m+2)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\frac{(m+\tau_1k_1+\tau_2k_2)(m+1+\tau_1k_1+\tau_2k_2)}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}(b_1)_{k_1}(b_2)_{k_2}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!}$$

This implies

$$= \frac{2m}{m+2}\left[\frac{\Gamma(m+3)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!}\right]$$

$$+\frac{2\tau_1(m+1)}{m+2}\left[\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=1,\\k_2=0}}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!}\right]$$

$$+\frac{2\tau_2(m+1)}{m+2}\left[\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=0,\\k_2=1}}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{(k_2-1)!}\right]$$

$$+\frac{m^2}{m+2}\left[\frac{\Gamma(m+3)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!}\right]$$

$$+\frac{\tau_1^2}{m+2}\left[\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=1\\k_2=0}}^{\infty}\frac{k_1\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!}\right]$$

$$+\frac{\tau_2^2}{(m+2)}\left[\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=0\\k_2=1}}^{\infty}\frac{k_2\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{(k_2-1)!}\right]$$

$$+\frac{2\tau_1\tau_2}{(m+2)}\left[\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=1\\k_2=1}}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{(k_2-1)!}\right] \qquad \ldots (20)$$

We express each summation of equation (20)in the R.H.S. is given as follows:

$$\frac{\partial^2}{\partial x^2}\left[x^2F_1^{(\tau_1,\tau_2)}(a,b_1,b_2;m+3;x,y)\right]$$

$$= 2F_1^{\tau_1,\tau_2}\left(a,b_1,b_2;m+3;x;y\right)+4x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}(a,b_1,b_2,m+3,x,y)$$

$$+x^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x^2}(a,b_1,b_2;m+3;x,y) \quad \ldots (21)$$

And $\frac{\partial^2}{\partial x^2}\left[x^2 F_1^{\tau_1,\tau_2}(a,b_1,b_2;m+3;x,y)\right]$

$$=\frac{\Gamma(m+3)}{\Gamma a}\sum_{k_1;k_2=0}^{\infty}\frac{(k_1+2)(k_1+1)\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!}$$

$$=\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_2=0\\k_1=1}}^{\infty}\frac{k_1\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!}$$

$$+3\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=1\\k_2=0}}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!}$$

$$+2\frac{\Gamma(m+3)}{\Gamma a}\sum_{k_1,k_2=0}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{k_2!} \quad \ldots(22)$$

Equating (21) & (22), we get

$$x^2\frac{\partial^2 F_1^{(\tau_1,\tau_2)}}{\partial x^2}+4x\frac{\partial F_1^{(\tau_1,\tau_2)}}{\partial x}=\frac{\Gamma(m+3)}{\Gamma a}\sum_{k_1=1,k_2=0}^{\infty}\frac{k_1\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!}$$

$$+3\frac{\Gamma(m+3)}{\Gamma a}\sum_{k_1=1,k_2=0}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!} \quad \ldots (23)$$

Similarly

$$y^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial y^2}+4y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}=\frac{\Gamma(m+3)}{\Gamma a}\sum_{k_1=0,k_2=1}^{\infty}\frac{k_2\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{(k_2-1)!}$$

$$+\frac{3\Gamma(m+3)}{\Gamma a}\sum_{k_1=0,k_2=1}^{\infty}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{(k_2-1)!} \quad \ldots (24)$$

Again considering $\frac{\partial^2}{\partial x\partial y}\left(xyF_1^{\tau_1,\tau_2}\right)=xy\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x\partial y}+y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}+x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}+F_1^{\tau_1,\tau_2}$ … (25)

Where $F_1^{\tau_1,\tau_2}=F_1^{\tau_1,\tau_2}(a,b_1,b_2;m+3;x,y)$

And $\frac{\partial^2}{\partial x\partial y}\left[\sum_{k_1,k_2=0}^{\infty}\frac{\Gamma(m+3)}{\Gamma a}\frac{\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1+1}}{k_1!}\frac{y^{k_2+1}}{k_2!}\right]$

$$= \frac{\Gamma(m+3)}{\Gamma a}\sum_{k_1,k_2=1}^{\infty}\frac{\Gamma a\left(a+\tau_1k_1+\tau_2k_2\right)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{(k_2-1)!}$$

$$+\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=1\\k_2=0}}^{\infty}\frac{\Gamma\left(a+\tau_1k_1+\tau_2k_2\right)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!}$$

$$+\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=0\\k_2=1}}^{\infty}\frac{\Gamma\left(a+\tau_1k_1+\tau_2k_2\right)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{(k_2-1)!}+F_1^{\tau_1,\tau_2} \quad \ldots (26)$$

Equating (25) & (26), we obtain

$$xy\frac{\partial^2F_1^{\tau_1,\tau_2}}{\partial x\partial y}+y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}+x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}=\sum_{k_1,k_2=1}^{\infty}\frac{\Gamma(m+3)}{\Gamma a}\frac{\Gamma\left(a+\tau_1k_1+\tau_2k_2\right)}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{(k_2-1)!}$$

$$+\sum_{\substack{k_1=1\\k_2=0}}^{\infty}\frac{\Gamma(m+3)}{\Gamma a}\frac{\Gamma\left(a+\tau_1k_1+\tau_2k_2\right)}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!}$$

$$+\sum_{\substack{k_1=0\\k_2=1}}^{\infty}\frac{\Gamma(m+3)}{\Gamma a}\frac{\Gamma\left(a+\tau_1k_1+\tau_2k_2\right)\left(b_1\right)_{k_1}\left(b_2\right)_{k_2}}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{(k_2-1)!} \quad \ldots (27)$$

Finally, we will take

$$\frac{\partial}{\partial x}\left(xF_1^{\tau_1,\tau_2}\right)=x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}+F_1^{\tau_1,\tau_2} \quad \ldots (28)$$

$$\frac{\partial}{\partial x}\left(xF_1^{\tau_1,\tau_2}\right)=\sum_{\substack{k_1=1\\k_2=0}}^{\infty}\frac{\Gamma(m+3)}{\Gamma a}\frac{\Gamma\left(a+\tau_1k_1+\tau_2k_2\right)\left(b_1\right)_{k_1}\left(b_2\right)_{k_2}}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{\left(k_1-1\right)!}\frac{y^{k_2}}{k_2!}+F_1^{\tau_1,\tau_2}(a,b_1,b_2,m+3,x,y) \quad \ldots (29)$$

Equating (28) & (29), we have

$$x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}=\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=1\\k_2=0}}^{\infty}\frac{\Gamma\left(a+\tau_1k_1+\tau_2k_2\right)\left(b_1\right)_{k_1}\left(b_2\right)_{k_2}}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{\left(k_1-1\right)!}\frac{y^{k_2}}{k_2!} \quad \ldots (30)$$

Similarly $$y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}=\frac{\Gamma(m+3)}{\Gamma a}\sum_{\substack{k_1=0\\k_2=1}}^{\infty}\frac{\Gamma\left(a+\tau_1k_1+\tau_2k_2\right)\left(b_1\right)_{k_1}\left(b_2\right)_{k_2}}{\Gamma\left(m+3+\tau_1k_1+\tau_2k_2\right)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{\left(k_2-1\right)!} \quad \ldots (31)$$

Using (30) & (31) in (23), (24) & (27), we get

$$x^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x^2}+x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}=\sum_{\substack{k_1=1\\k_2=0}}^{\infty}\frac{\Gamma(m+3)}{\Gamma a}.\frac{k_1\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{k_2!}\ \ldots (32)$$

And

$$y^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x^2}+y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}=\sum_{\substack{k_1=1\\k_2=0}}^{\infty}\frac{\Gamma(m+3)}{\Gamma a}.\frac{k_2\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{k_1!}\frac{y^{k_2}}{(k_2-1)!}\ \ldots (33)$$

$$xy\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x\partial y}=\sum_{\substack{k_1=1\\k_2=1}}^{\infty}\frac{\Gamma(m+3)}{\Gamma a}.\frac{k_2\Gamma(a+\tau_1k_1+\tau_2k_2)(b_1)_{k_1}(b_2)_{k_2}}{\Gamma(m+3+\tau_1k_1+\tau_2k_2)}\frac{x^{k_1}}{(k_1-1)!}\frac{y^{k_2}}{(k_2-1)!}\ \ldots (34)$$

Now applying (32), (33), (34), (30) & (31) in (20), we get

$$U=\frac{2m}{(m+2)}F_1^{\tau_1,\tau_2}+\frac{2\tau_1(m+1)}{(m+2)}\left[x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}\right]+\frac{2\tau_2(m+1)}{(m+2)}\left[y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}\right]$$

$$+\frac{m^2}{(m+2)}F_1^{\tau_1,\tau_2}+\frac{\tau_1^2}{(m+2)}\left[x^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x^2}+x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}\right]$$

$$+\frac{\tau_2^2}{(m+2)}\left[y^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial y^2}+y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}\right]+\frac{2\tau_1\tau_2}{(m+2)}\left[xy\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x\partial y}\right]\qquad \ldots (35)$$

Using (35) in (19), we get

$$=(m+1)F_1^{\tau_1,\tau_2}(a,b_1,b_2;m+2;x,y)-F_1^{\tau_1,\tau_2}(a,b_1,b_2;m+2;x,y)$$

$$=\frac{\tau_1^2}{(m+2)}x^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial x^2}+\frac{\tau_2^2}{(m+2)}y^2\frac{\partial^2 F_1^{\tau_1,\tau_2}}{\partial y^2}+\frac{2\tau_1\tau_2}{(m+2)}xy\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}$$

$$+\frac{\tau_1}{(m+2)}\left[\tau_1+2(m+1)\right]x\frac{\partial F_1^{\tau_1,\tau_2}}{\partial x}+\frac{\tau_2}{(m+2)}\left[\tau_2+2(m+1)\right]y\frac{\partial F_1^{\tau_1,\tau_2}}{\partial y}+mF_1^{\tau_1,\tau_2}$$

Which prove the result (5.1)

(i) On taking $n=2$ ingeneralized Lauricella function $F_D^{(n)(\tau_1,\ldots,\tau_n)}(a;b_1,\ldots b_n;c;x_1,\ldots,x_n)$, we obtain all the result of equations 10, 11 and theorem 3.1, 3.2, 3.3, 3.4, 4.1, 4.2involving τ-genearlized appell function.

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