Information **S**ciences and **C**omputing
Volume 2014, Number 1, Article ID ISC251013, 03 pages
Available online at http://www.infoscicomp.com/

Research Article

On the Riemann Zeta Function at Even Integers

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Received 12 October 2013; Accepted 19 December 2013

Abstract: Without the explicit participation of Bernoulli polynomials $\phi_n(x)$, Balanzario [1] exhibits an elementary method to determine $\zeta(2n)$, n = 1, 2,… Here we show the connection between his polynomials and the $\phi_n(x)$.

Keywords: Riemann zeta function; Bernoulli polynomials.

1. Introduction

In [1] the principal aim is to calculate the Riemann zeta function [2, 3] at even integers, with the following result:

$$\zeta(2n) = -\frac{(-4)^n}{4^n-2}\ (2\,\pi)^{2\,n}\ p_{2n+1}\left(\frac{1}{2}\right) \quad , \tag{1}$$

where

$$p_2(x) = \frac{x^2}{2} \ ,\ p_3(x) = \frac{x^3}{6} \ ,\ p_4(x) = \frac{x^4}{24} - \frac{x^2}{48} \ ,$$

$$p_5(x) = \frac{x^5}{120} - \frac{x^3}{144} \ ,\ p_6(x) = \frac{x^6}{720} - \frac{x^4}{576} + \frac{x^2}{11520} \ , \tag{2}$$

$$p_7(x) = \frac{x^7}{5040} - \frac{x^5}{2880} + \frac{7\,x^3}{34560} \ , \ldots$$

These polynomials can be generated with $p_2 = \frac{x^2}{2}$ and the recurrence relations:

$$p_{2k+1}(x) = \int^x p_{2k}\, d\eta \ ,$$

$$p_{2k+2}(x) = \int^x p_{2k+1}\, d\eta \ - \ x^2\, p_{2k+1}\left(\frac{1}{2}\right) \ , \tag{3}$$

without constants of integration.

Thus with (1) and (2) is easy to obtain that $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, $\zeta(6)=\frac{\pi^6}{945}$, ... , in harmony with the values reported in the literature [2, 3]. It would be interesting to have a closed expression for $\zeta(2n+1)$ [4]. In the next Section, we show how to write (1) and (2) in terms of the polynomials and numbers of Bernoulli.

2. Balanzario's polynomials

We have the Bernoulli polynomials [5]:

$$\phi_1(x)=x \quad , \quad \phi_2(x)=x^2-x \quad , \quad \phi_3(x)=x^3-\frac{3}{2}x^2+\frac{x}{2} \quad ,$$
$$\phi_4(x)=x^4-2x^3+x^2 \quad , \quad \phi_5(x)=x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{x}{6} \quad , \tag{4}$$
$$\phi_6(x)=x^6-3x^5+\frac{5}{2}x^4-\frac{x^2}{2} \quad ,....$$

with the properties:

$$\phi_1\left(\frac{1}{2}\right)=\frac{1}{2} \quad , \quad \phi_j(1)=0 \quad , \quad j=2,3,.....$$
$$\phi_{2r+1}\left(\frac{1}{2}\right)=0 \quad , \quad r=1,2,.... \quad ; \quad \phi_k(0)=0 \quad \forall\ k\,. \tag{5}$$

Then the polynomials (2) can be written in terms of (4), for example,

$$p_3(x)=\frac{1}{24}\left[\phi_3(2x)-4\,\phi_3(x)+\phi_1(x)\right] \quad ,$$
$$p_5(x)=\frac{1}{5760}\left[3\,\phi_5(2x)-48\,\phi_5(x)-7\phi_1(x)\right] \quad , \quad \text{etc.} \tag{6}$$

therefore:

$$p_3\left(\frac{1}{2}\right)=\frac{1}{48} \quad , \quad p_5\left(\frac{1}{2}\right)=-\frac{7}{11520} \quad , \tag{7}$$

thus (1) gives $\zeta(2)=\frac{\pi^2}{6}$ and $\zeta(4)=\frac{\pi^4}{90}$.

Besides, it is possible to prove that:

$$p_{2k+1}\left(\frac{1}{2}\right)=-\frac{1}{2}\tilde{B}_{2n}\left(\frac{1}{2}\right)=(-1)^n\frac{1-2^{2n-1}}{2^{2n}(2n)!}\tilde{B}_{2n} \quad , \tag{8}$$

where $\tilde{B}_{2n}$ are the Faulhaber [6]–Bernoulli [7] numbers:

$$\tilde{B}_2=\frac{1}{6} \quad , \quad \tilde{B}_4=\tilde{B}_8=\frac{1}{30} \quad , \quad \tilde{B}_6=\frac{1}{42} \quad ,..... \tag{9}$$

with the Bernoulli-like polynomials [5]:

$\tilde{B}_1(x) = x \quad , \; \tilde{B}_2(x) = \frac{1}{6}\,(3\,x^2 - 1) \;\; , \quad \tilde{B}_3(x) = \frac{1}{6}\,(x^3 - x)\,,$

$$\tilde{B}_4(x) = \frac{1}{360}\,(15\,x^4 - 30\,x^2 + 7) \;\; , \quad \tilde{B}_5(x) = \frac{1}{360}\,(3\,x^5 - 10\,x^3 + 7\,x) \;\; , \qquad (10)$$

$\tilde{B}_6(x) = \frac{1}{15120}\,(21\,x^6 - 105\,x^4 + 147\,x^2 - 31) \quad ,\ldots.$

Then

$$\tilde{B}_2\left(\frac{1}{2}\right) = -\,\frac{1}{24} \;, \qquad \tilde{B}_4\left(\frac{1}{2}\right) = \frac{7}{5760} \;, \qquad (11)$$

in accordance with (7) and (8). If into (1) we employ (8) appears the known relation [2, 8]:

$$\zeta(2n) = \frac{2^{2n-1}}{(2n)!}\,\pi^{2n}\;\tilde{B}_{2n}\,.$$

The expressions (6) and (8) show the relationship between the polynomials of Balanzario and Bernoulli.

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