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Research Article

Some Fixed Point Theorems in Symmetric Spaces

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Abstract: In this note we extend some fixed point theorems in metric spaces to symmetric spaces.

Keywords: Symmetric spaces, metric spaces, $(E.A)$ property, compatible mappings, coincidence points.

1. Introduction

Symmetric spaces are spaces which result when the triangular inequality in a metric space is dropped. The dropping of the triangular inequality results in the loss of the concept of closeness between distinct points and so precludes the possibility of obtaining fixed points by successive approximations. Given a self map f on a symmetric space X, a suitable contractive condition has to be imposed on f to ensure that iterates come close and also suitable conditions on (X,d) have to be imposed to ensure that sequences with closely packed points do converge. In classical fixed point theory, completeness of the metric space (X,d) ensures this. In the case of symmetric spaces we impose conditions such as $(W3)$, $(W4)$, $(H.E)$, $(C.C)$ and $(E.A)$ to ensure the good behavior of sequences intended to converge to fixed points. The various conditions listed above have been in vogue in the fixed point theory literature [see 1, 2, 3, 4, 6, 8, 9] for quite some time. Some special cases and applications of symmetric spaces are discussed by many authors [see 7, 9, 10, 11].

2. Preliminaries on Symmetric Spaces

A *symmetric* on a set X is a function $d: X\times X \to [0,\infty)$ satisfying the following conditions:

I. $d(x,y)=0$, *if and only if* $x=y$ *for* $x,y\in X$

II. $d(x,y)=d(y,x)$, *for all* $x,y\in X$

If (X,d) is a symmetric space, Let $B(x,\in)$ denotes $\{y\in X; d(x,y)<\in\}$. A topology $\tau(d)$ on X *can be* defined as follows: $u\in\tau(d)$ if and only if, to each $x\in u$, there corresponds a positive number $\in(x)$ such that $B(x,\in(x))\subseteq u$. A subset S of X is a neighborhood of x of X if there exists $u\in\tau(d)$ such that $x\in u\subset S$. A symmetric d is a *semi-metric* if for each $x\in X$ and each $\in>0$, $B(x,\in)$ is a neighborhood of X in the topology $\tau(d)$. A symmetric space (X,d) is not necessarily Hausdorff. Consequently limits of sequences need not be unique. We consider the following conditions on a symmetric space (X,d) (see [2, 5]).

$(W3)$: *For a sequence* $\{x_n\}$ *in* X *and* $x,y\in X$,

$\lim_{n\to\infty} d(x_n,x)=0$ *and* $\lim_{n\to\infty} d(x_n,y)=0$ *implies* $x=y$

Thus $(W3)$ could be described as **"Uniqueness of Limits"**.

$(W4)$: *For a sequence* $\{x_n\},\{y_n\}$ *in* X *and* $x\in X$,

$\lim_{n\to\infty} d(x_n,x)=0$ *and* $\lim_{n\to\infty} d(y_n,x_n)=0$

implies $\lim_{n\to\infty} d(y_n,x)=0$

$(W4)$ could be called **"limit-sharing by equivalent sequences"**.

Also the following axiom can be found in [7].

$(H.E)$: *For a sequence* $\{x_n\},\{y_n\}$ *in* X *and* $x\in X$,

$\lim_{n\to\infty} d(x_n,x)=0$ *and* $\lim_{n\to\infty} d(y_n,x)=0$

implies $\lim_{n\to\infty} d(x_n,y_n)=0$

$(H.E)$ Could be called "**Equivalence of sequences which have identical limits**".

Now, we add a new axiom which is related to the continuity of the symmetric 'd'.

$(C.C)$: *For a sequence* $\{x_n\}$ *in* X *and* $x, y \in X$,

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ implies } d(x_n, y) = d(x, y)$$

$(C.C)$ Could be called "**coordinate-wise Continuity of distance**".

Note that if d is a metric, then all these conditions are satisfied.

Note: we write $\lim_{n\to\infty} a_n = A$ for a sequence $\{a_n\}$ and a subset A of a symmetric space if $a = \lim_{n\to\infty} a_n$ for every $a \in A$ and A is the set of all such points. By an abuse of notation, we sometimes write $a = \lim_{n\to\infty} a_n$ instead of $a \in \lim_{n\to\infty} a_n$.

Definition 2.1: (See [8]) Let $S, T : X \to X$. The pair (S, T) satisfies property $(E.A)$ if there exists a sequence $\{x_n\}$ *in* X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \in X$. $(E.A)$ Could be called "**Coincidence of** S *and* T **on a sequence**".

Remark: It is easy to check that $(W4) \Rightarrow (W3)$ and $(C.C) \Rightarrow (W3)$.

3. Main Results

Theorem 3.1: Let (X, d) be a symmetric (semi-metric) space that satisfies $(C.C)$ and $(H.E)$ and let A, B, S and T be self-mappings of X such that

(1) $AX \subset TX$ *and* $BX \subset SX$

(2) The pair (B, T) satisfies property $(E.A)$ (resp., (A, S) satisfies property $(E.A)$)

(3) For any $x, y \in X$,
$d(Ax, By) \le \max\{kd(Sx, Ax), kd(Ty, By), kd(Sx, Ty)\}$ for a constant k satisfying $0 < k < 1$

(4) SX is closed subset of X

Then there exists $u, w \in X$ such that $Au = Su = Bw = Tw$

Proof: From (2), there exist a sequence $\{x_n\}$ in X, and a point t in X such that $\lim_{n\to\infty} d(Tx_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$

From (1), there exists a sequence $\{y_n\}$ in X, such that $Bx_n = Sy_n$. Now $d(Sy_n, Tx_n) \to 0$ *by* $(H.E)$. From (4), there exists a point u in X such that $Su = t$. From (3), we have

$$0 \le d(Au, Bx_n) \le \max\{kd(Su, Au), kd(Tx_n, Bx_n), kd(Su, Tx_n)\}$$

Now $\lim_{n\to\infty} \max\{kd(Su, Au), kd(Tx_n, Bx_n), kd(Su, Tx_n)\}$

$= \max\{kd(Su, Au), k\lim_{n\to\infty} d(Tx_n, Bx_n), k\lim_{n\to\infty} d(Su, Tx_n)\}$

$= \max\{kd(Su, Au), 0, 0\}$

$= kd(Su, Au) = kd(t, Au)$

By $(C.C)$, $\lim_{n\to\infty} d(Au, Bx_n) = d(Au, t) = 0.$ Thus $Au = Su$

Since $AX \subset TX$ there exists $w \in X$ such that $Au = Tw$

We shall now prove that $Tw = Bw$

From (3) we have $d(Au, Bw) \le \max\{kd(Su, Au), kd(Tw, Bw), d(Su, Tw)\}$
$= \max\{0, kd(Au, Bw), 0\}$
$= kd(Au, Bw)$

Since $0 < k < 1$, we conclude that $d(Au, Bw) = 0.$
$\therefore Au = Bw.$ This completes the proof of the theorem.

For existence of common fixed point of all the four self maps we need the additional property of weak compatibility of the pairs $(A,S),(B,T)$. Recall that for self mappings f and g of a set, the pair (f,g) is said to be weakly compatible [8] if $fgx = gfx$, whenever $fx = gx$. It is Obvious that commuting mappings are weakly compatible.

Theorem 3.2: Let (X, d) be a symmetric (semi-metric) space that satisfies $(C.C)$ and $(H.E)$ and let A, B, S and T be self-mappings of X such that

(1) $AX \subset TX$ *and* $BX \subset SX$
(2) The pair (B,T) satisfies property $(E.A)$
(3) The Pairs (A,S) and (B,T) are weakly compatible
(4) For any $x, y \in X$, $d(Ax, By) \le \max\{kd(Sx, Ax), kd(Ty, By), kd(Sx, Ty)\}$
For some constant k such that $0 < k < 1$
(5) SX is closed in X.

Then A, B, S and T have a unique common fixed point in X.

Proof: From theorem 3.1, there exist $u, w \in X$ such that $Au = Su = Tw = Bw.$

From (3), $ASu = SAu, AAu = ASu = SAu = SSu$ *and* $BTw = TBw = TTw = BBw.$

From (4), we have

$$d(Au, AAu) = d(AAu, Bw)$$
$$\leq \max\{kd(SAu, AAu), kd(Tw, Bw), kd(SAu, Tw)\}$$
$$= \max\{o, o, kd(AAu, Tw)\}$$
$$= kd(AAu, Au)$$

Since $0 < k < 1$, it follows that $d(AAu, Au) = 0$
$\therefore Au$ is a fixed point of S.

$$d(Bw, BBw) = d(Au, BBw)$$
$$\leq \max\{kd(Su, Au), kd(TBw, BBw), kd(Su, TBw)\}$$
$$= \max\{o, o, kd(Su, TBw)\}$$
$$= kd(Bw, TBw) = kd(Bw, BBw)$$

Since $0 < k < 1$, it follows that $d(Bw, BBw) = 0$
$\therefore Bw$ is a fixed point of B.

$Bw = B(Bw) = BBw = T(Bw)$ shows that Bw is a fixed point of T.
$\therefore Au = Bw$ is a fixed point of A, B, S and T.

Uniqueness: Let z, w be any fixed point of A, B, S and T. Then

$$d(z, w) = d(Az, Bw)$$
$$\leq k \max\{d(Sz, Az), d(Tw, Bw), d(Sz, Tw)$$
$$= d(z, w)$$
Which implies $d(z, w) = 0$ and $z = w$

Example 3.3: We give an example of a symmetric space which has the property $(H.E)$ but does not have the property $(C.C)$.

Let $X = \{0,1,2,3...\}$. Define $d(n,0) = \frac{1}{n} = d(0,n)$ for $n = 1,2,3...$
$d(0,0) = 0, d(n,n) = 0$ for $n = 1,2,3...$
And $d(m,n) = \frac{1}{m+n}$ for $m > 0, n > 0$.
Let $x_n = n$, for $n = 1,2,3...$
Then $d(x_n, 4) = \frac{1}{n+4}$ *and* $d(0,4) = \frac{1}{4}$
$d(x_n, 4) \not\to d(0,4)$. This shows that (X, d) does not have $(C.C)$.

We shall now check that (X, d) satisfies $(H.E)$.

Suppose $d(x_n, x) \to 0$ and $d(y_n, x) \to 0$.

To establish that (X, d) has $(H.E)$, we need to show that $d(x_n, y_n) \to 0$.

Now $d(x_n, x) = 0 \; if \; x_n = x = 0$

$$= \frac{1}{x_n + x} \quad \text{otherwise.}$$

Also $d(y_n, x) = 0 \; if \; y_n = y = 0$

$$= \frac{1}{y_n + x} \quad \text{otherwise.}$$

Choose a positive integer k so big that $k > x$ and $\frac{1}{2k - 2x} < \in$

Since $d(x_n, x) \to 0$ as $n \to \infty$ and $d(y_n, x) \to 0$ as $n \to \infty$ there exists a positive integer n_0 such that $d(x_n, x) < \frac{1}{2k}$ and $d(y_n, x) < \frac{1}{2k}$ if $n \geq n_0$.

We claim that $d(x_n, y_n) < \in$ if $n \geq n_0$

To show this we consider four cases,

Case (i) Suppose $n > n_0$, $x_n = x = 0 \; and \; y_n = x = 0$ then

$d(x_n, y_n) = d(0,0) = 0 < \in$

Case (ii) Suppose $n > n_0$, $x_n = x = 0 \; and \; y_n \neq 0$ then

$d(x_n, y_n) = d(0, y_n) = \frac{1}{y_n} < \frac{1}{2k} < \in$

$d(x, y_n) < \frac{1}{2k} < \in$ for all $n \geq n_0$

Case (iii) Suppose $n \geq n_0$, $x_n \neq 0$ and $y_n = x = 0$

This case can be deal with exactly as in case (ii).

Case (iv) Suppose $n > n_0$ and one of x_n, x and one of y_n, x are different from 0.

Then $d(x_n, x) = \frac{1}{x_n + x}$ and $d(y_n, x) = \frac{1}{y_n + x}$

$\therefore \frac{1}{x_n + x} \leq \frac{1}{2k}$

$\Rightarrow 2k \leq x_n + x$

$\Rightarrow 2k - x \leq x_n$

Similarly $2k - x \leq y_n$

By adding $4k - 2x \leq x_n + y_n$

$$\frac{1}{x_n + y_n} \le \frac{1}{4k - 2x} \le \frac{1}{2k - 2x} < \in$$

Thus $d(x_n, y_n) < \in$ if $n \ge n_0$ in all the cases.

Example 3.4: Let $X = [0,1]$ and $d(x, y) = (x - y)^2$. Define self-mappings A, B, S and T by $Ax = Bx = \left(\frac{1}{2}\right)x$ *and* $Sx = Tx = x$ for all $x \in X$ .Then, we have the following:

(1) (X, d) is a symmetric space satisfying the properties $(H.E)$ &$(C.C)$

(2) $AX \subset TX$ *and* $BX \subset SX$,

(3) The pair (B, T) satisfies property $(E.A)$ for the sequence $x_n = \frac{1}{n}$, $n = 1, 2, 3...$,

(4) The pairs (A, S) *and* (B, T) are weakly compatible,

(5) For $x, y \in X$ $(x \ne y)$ $d(Ax, By) \le \max\{d(Sx, Ax), d(Ty, By), d(Sx, Ty)\}$,

(6) SX *is a* d - closed subset of X ,

(7) $A0 = B0 = S0 = T0 = 0$.

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