Information **S**ciences and **C**omputing
Volume 2016, Article ID ISC671116, 12 pages
Available online at http://www.infoscicomp.com/

Research Article

On Inner Product Quasilinear Spaces and Hilbert Quasilinear Spaces

Hacer Bozkurt[a] and Yılmaz Yilmaz[b]

[a]Department of Mathematics, Batman University, 72100, Batman, Turkey
[b]Department of Mathematics, İnönü University, 44280, Malatya, Turkey

Corresponding author: Hacer Bozkurt; E-mail: hacer.bozkurt@batman.edu.tr

Received 09 November 2016; Accepted 20 December 2016

Abstract: In this paper, we state some properties of inner product quasilinear spaces. Moreover we introduce the concepts of inner product Ω-space, Hilbert Ω-space and investigate some related theorems. Also, we establish some differences of inner product quasilinear spaces from the inner product (linear) spaces.

Keywords: Quasilinear space; Inner product quasilinear space; Hilbert quasilinear space; Orthogonality; Orthonormality.

1. Introduction and Known Results about Quasilinear Spaces and Normed Quasilinear Spaces

The notion of quasilinear space was introduced by Aseev in 1986, [1]. Generally in [1], he stated properties which are quasilinear counterparts of some results in linear functional analysis. This work has led to a lot of authors to introduce new results on set-valued analysis, [2], set-differential equations [3], fuzzy normed spaces [4], etc. Also, recently, the concepts of topological quasilinear spaces as a generalization of the notion of normed quasilinear spaces defined by Aseev, proper quasilinear spaces introduced in [5], [6], [7] and [8].

Because of the theory of inner product space and Hilbert spaces play a fundamental role in functional analysis and its applications, we generalized the notion of linear inner product spaces to the quasilinear context in [9]. Following the definition of inner product quasilinear space introduced by Y. Yılmaz, we study some results. While working on this new concept, we noticed that there were some differences related to analysis as different from classical case and we obtained some significant results on inner product quasilinear spaces [10] and [11].

In this study, we prove that the norm of every inner product quasilinear space may not satisfy the parallelogram law. Also introduce the concept of inner product Ω-space and Hilbert Ω-space. Some immediate theorems are also proved. In this paper we aim to give a contribution to the studies on inner product quasilinear spaces by introducing the new results.

In this section, we will give some definitions and preliminary results given by Aseev [1].

Definition 1: *[1] A set X is called a quasilinear space if a partial order relation "$\leqslant$", an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in R$:*

(1) $x \leq x$,
(2) $x \leq z$ if $x \leq y$ and $y \leq z$,
(3) $x = y$ if $x \leq y$ and $y \leq x$,
(4) $x + y = y + x$,
(5) $x + (y + z) = (x + y) + z$,
(6) there exists an element $\theta \in X$ such that $x + \theta = x$,
(7) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$,
(8) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$,
(9) $1 \cdot x = x$,
(10) $0 \cdot x = \theta$,
(11) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$,
(12) $x + z \leq y + v$ if $x \leq y$ and $z \leq v$ x+z,
(13) $\alpha \cdot x \leq \alpha \cdot y$ if $x \leq y$.

A linear space is a quasilinear space with the partial order relation "=". The most popular example which is not a linear space is the set of all closed intervals of real numbers with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A + B = \{a + b : a \in A, b \in B\}$$

and the real-scalar multiplication

$$\lambda \cdot A = \{\lambda \cdot a : a \in A\}.$$

We denote this set by $\Omega_C(R)$.

Another one is $\Omega(R)$, which is the set of all compact subsets of real numbers. By a slight modification of algebraic sum operation (with closure) such as

$$A + B = \{a + b : a \in A, b \in B\},$$

with the same real-scalar multiplication defined above and the inclusion relation we can give as example the nonlinear quasilinear space $\Omega(E)$ and $\Omega_C(E)$ the families of all nonempty closed bounded and convex closed bounded subsets of the normed linear space E , respectively.

Lemma 2: *[1] Suppose that every element x in a quasilinear space X has an inverse element $x^{'} \in X$. Then the partial order relation in X is determined by equality, the distributivity condition in (11) holds, and consequently, X is a linear space [1].*

Let X be a quasilinear space and $Y \subseteq X$. Then Y is called a subspace of X whenever Y is a quasilinear space with the same partial order and the restriction to Y of the operations on X. One can easily prove the following theorem by using the axioms of to be a quasilinear space. It is quite similar to its linear space analogue.

Theorem 3: *[6] Let X be a quasilinear space and $Y \subseteq X$. Then $\alpha \cdot x + \beta \cdot y \in Y$ for every, $x, y \in Y$ and $\alpha, \beta \in R$.*

Let X be a quasilinear space. An element $x \in X$ is said to be symmetric if $-x = x$, where $-x = (-1) \cdot x$, and X_d denotes the set of all symmetric elements. θ denotes the unit element of addition operation in X, and it is minimal element, i.e., $x = \theta$ if $x \leq \theta$. An element $x^{'}$ is called inverse of x if $x + x^{'}$. If the inverse element exists, then it is unique and $x^{'} = -x$ in this case. Sometimes $x^{'}$ may not be exist but $-x$ is always meaningful in quasilinear spaces. An element x possessing inverse is called regular, otherwise is called singular. Now, X_r and X_s stand for the sets of all regular and singular elements in X, respectively. Further, X_r, X_d and $X_s \cup \{0\}$ are subspaces of X and they are called regular, symmetric and singular subspaces of X, respectively [9]. It is easy from the definitions that $X_d \subset X_s$ and $X = X_r \cup X_s$. For a singular element x we should note that $x - x \neq \theta$. In a linear quasilinear space, that is, in a linear space, there are no singular elements. Also, in a quasilinear space X, it is obvious that an element x is regular if and only if $x - x = \theta$.

Proposition 5: *[6] In a quasilinear space X every regular element is minimal.*

Definition 6: *[1] Let X be a quasilinear space. A function $\|\cdot\|_X : X \to R$ is called a norm if the following conditions satisfied:*

(14) $\|x\|_X > 0$ if $x \neq \theta$,
(15) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$,
(16) $\|\alpha \cdot x\|_X = \alpha \cdot \|x\|_X$,
(17) if $x \leq y$, then $\|x\|_X \leq \|y\|_X$,
(18) if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \leq y + x_\varepsilon$ and $\|x_\varepsilon\|_X \leq \varepsilon$ then $x \leq y$.

A quasilinear space X with a norm defined on it, is called normed quasilinear space.

It follows from Lemma 2 that if every $x \in X$ has inverse element $x^{'} \in X$, then the concept of normed quasilinear space coincides with the concept of normed linear space, [1].

Let X be a normed quasilinear space. Hausdorff metric or norm metric on X is defined by the equality

$$h_X(x,y) = \inf\left\{r \geq 0 : x \leq y + a_1^r, y \leq x + a_2^r \, and \left\|a_i^r\right\| \leq r, i = 1,2\right\}.$$

Since $x \leq y + (x-y)$ and $y \leq x + (y-x)$, the quantity $h_X(x,y)$ is well-defined and satisfies all of the metric axioms, also

$$h_X(x,y) \leq \|x-y\|_X$$

for all elements $x, y \in X$.

Lemma 7: *[1] The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is continuous function respect to the Hausdorff metric.*

Example 8: *[1] Let E be a normed linear space. A norm on $\Omega(E)$ is defined by*

$$\|A\|_{\Omega(E)} = \sup_{a \in A} \|a\|_E .$$

$\Omega_C(E)$ is a also a normed quasilinear space with the norm above. In this case, the Hausdorff metric on $\Omega(E)$ and $\Omega_C(E)$ is defined as usual:

$$h_\Omega(A,B) = \inf\{r \geq 0 : A \subset B + S_r(\theta), B \subset A + S_r(\theta)\},$$

where $S_r(\theta)$ is the closed ball of radius r and centered at $\theta \in X$.

Definition 9: *[8] Let X be a quasilinear space, $M \subseteq X$ and $x \in M$. The set*

$$F_x^M = \{z \in M_r : z \leq x\}$$

is called floor in M of x. In the case of $M = X$ it is called only floor of x and written briefly F_x instead of F_x^X .

Floor of an element x in linear spaces is $\{x\}$. Therefore, it is nothing to discuss the notion of floor of an element in a linear space.

Definition 10: *[9] Let X be a quasilinear space and $M \subseteq X$. Then the union set*

$$\bigcup_{x \in M} F_x^M$$

is called floor of M and is denoted by F_M. In the case of $M = X$, F_X is called floor of the quasilinear space X .

On the other hand, the set

$$F_M{}^X = \bigcup_{x \in M} F_x{}^X$$

is called floor in X of M and is denoted by $F_M{}^X$.

Definition 11: *[9] A quasilinear space X is called solid floored quasilinear space whenever $\sup F_y$ exists for every $y \in X$ and*

$$y = \sup F_y = \sup\{z \in X_r : z \leq y\}.$$

Otherwise, X is called non-solid floored quasilinear space.

Example 12: *[9] For any normed linear space E, $\Omega(E)$ and $\Omega_C(E)$ are solid floored quasilinear spaces.*

On the other hand, it is clear that $(\Omega_C(R))_s \cup \{0\}$ is non-solid floored quasilinear space. For example,

$$\sup\{x : x \in ((\Omega_C(R))_s \cup \{0\})_r , x \subseteq y\} = \{0\} \neq y$$

for element $y = [-2,3] \in (\Omega_C(R))_s \cup \{0\}$. Also, no there exists any element x such that $x \subseteq z$ for $z = [1,3] \in (\Omega_C(R))_s \cup \{0\}$.

2. Inner Product Quasilinear Spaces

Let X be a quasilinear space. Consolidation of floor of X is the smallest solid-floored quasilinear space $\hat{X}$ containing X, that is, if there exists another solid-floored quasilinear space Y containing X then $\hat{X} \subseteq Y$.

Clearly, $\hat{X} = X$ for some solid-floored quasilinear space X . For a quasilinear space X , the set

$F_y{}^{\hat{X}} = \{z \in (\hat{X})_r : z \leq y\}$ is the floor of y in $\hat{X}$ [9].

Definition 13: *[9] Let X be a quasilinear space. A mapping $\langle \cdot,\cdot \rangle : X \times X : \Omega(R)$ is called an inner product on X if the following conditions are satisfied for any $x, y, z \in X$ and $\alpha \in R$:*

(19) if $x, y \in X_r$ then $\langle x, y \rangle \in (\Omega_C(R))_r \equiv R$,

(20) $\langle x + y, z \rangle \subseteq \langle x, z \rangle + \langle y, z \rangle$,

(21) $\langle \alpha \cdot x, y \rangle = \alpha \cdot \langle x, y \rangle$,

(22) $\langle x, y \rangle = \langle y, x \rangle$,

(23) $\langle x, x \rangle \geq 0$ for $x \in X_r$ and $\langle x, x \rangle = 0 \Leftrightarrow x = \theta$,

(24) $\left\| \langle x, y \rangle \right\|_{\Omega(R)} = \sup\left\{ \left\| \langle a,b \rangle \right\|_{\Omega(R)} : a \in F_x^{\tilde{X}}, b \in F_y^{\hat{X}} \right\}$

(25) if $x \leq y$ and $u \leq v$ then $\langle x,u \rangle \subseteq \langle y,v \rangle$,

(26) if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \leq y + x_\varepsilon$ and $\langle x_\varepsilon, x_\varepsilon \rangle \subseteq S_\varepsilon(\theta)$ then $x \leq y$.

A quasilinear space with an inner product is called inner product quasilinear space.

Remark 14: *If X is a linear space, then above conditions are determined by conditions of the real inner product spaces. Also, a regular subspace X_r of an inner product quasilinear space X is a (linear) inner product space with the same inner product.*

Example 15: *[9] One can easily seen that $\Omega_C(R)$, the space of closed real intervals, is an inner product quasilinear space with inner product defined by*

$$\langle A,B \rangle = \{ a \cdot b : a \in A, b \in B \}. \qquad (1)$$

Lemma 16: *(Schwarz Inequality) For any two elements x and y of an inner product quasilinear space X, we have*

$$\left\| \langle x, y \rangle \right\|_{\Omega(R)} \leq \|x\|_X \|y\|_X .$$

Proof: Let X be an inner product quasilinear space. From (24) and Remark 14, we obtain

$$\begin{aligned} \left\| \langle x, y \rangle \right\|_{\Omega(R)} &= \sup\left\{ \left\| \langle a,b \rangle \right\|_{\Omega(R)} : a \in F_x^{\hat{X}}, b \in F_y^{\hat{X}} \right\} \\ &= \sup\left\{ \left| \langle a,b \rangle \right| : a \in F_x^{\hat{X}}, b \in F_y^{\hat{X}} \right\} \\ &\leq \sup\left\{ \|a\|_{\hat{X}} \|b\|_X : a \in F_x^{\hat{X}}, b \in F_y^{\hat{X}} \right\} \\ &\leq \sup\left\{ \|a\|_{\hat{X}} : a \in F_x^{\hat{X}} \right\} \sup\left\{ \|b\|_{\hat{X}} : b \in F_y^{\hat{X}} \right\} \\ &= \|a\|_X \|b\|_X \end{aligned}$$

for all $x, y \in X$.

Every inner product quasilinear space X is a normed quasilinear space with the norm defined by

$$\|x\| = \sqrt{\left\| \langle x, x \rangle \right\|_{\Omega(R)}}$$

for every $x \in X$. This norm is called inner product norm. The classical norm of $\Omega_C(R)$ is generated by the above inner product [9].

Now we are going to give an example, which is more general than Example 15.

Example 17: *The space* $\Omega_C\left(R^n\right)$ *of all compact and convex subsets of* R^n, *with the inner product defined by*

$$\langle A,B\rangle = \left\{\langle a,b\rangle_{R^n} : a\in A, b\in B\right\}$$

is an inner product quasilinear space.

Remark 18: *Different from the linear functional analysis, a norm on an inner product quasilinear space does not have to satisfy the parallelogram equality.*

Example 19: *We know that* $\Omega_C(R)$ *is an inner product quasilinear space with inner product (1). A norm on this inner product is*

$$\|A\| = \sqrt{\|\langle A,A\rangle\|_{\Omega(R)}} = \sqrt{\left\|\left\{a^2 : a\in A\right\}\right\|_{\Omega(R)}} = \sup_{a\in A}|a|$$

for every $A\in\Omega_C(R)$. Further, for $A = B = [0,1]\in\Omega_C(R)$, we have

$$\|A\| = 1, \|B\| = 1 \text{ and } \|A+B\| = 2, \|A-B\| = 1.$$

But,

$$\|A+B\|^2 + \|A-B\|^2 \neq 2\left(\|A\|^2 + \|B\|^2\right).$$

Theorem 20: *Regular subspace of an inner product quasilinear space has always satisfy the parallelogram law.*

Proposition 21: *[9]* $x_n \to x$ *and* $y_n \to y$ *in an inner product quasilinear space then* $\langle x_n, y_n\rangle \to \langle x,y\rangle$.

Lemma 22: *Let* X *be an inner product quasilinear space. If* $x+y\in X_r$ *for every* $x,y\in X$, *then* $x\in X_r$ *and* $y\in X_r$.

Proof: Let $x+y\in X_r$ and $x\notin X_r$. In this case, there exists an $z\in X_r$ such that $(x+y)+z=0$. This shows that x has an inverse, because $x+(y+z)=0$. This is contradiction with our acceptance.

3. Hilbert Quasilinear Spaces

In this section, we introduce the notions of Ω-space and Hilbert Ω-space. Then we obtain some results related to these new concepts. Moreover, in this part, we study some results on inner product quasilinear spaces.

Definition 23: *[1] An inner product quasilinear space is called an inner product Ω-space, if there exists an element* $B_X \neq \theta$ *such that* if $\|x\| \le \|B_X\|$ then $x \le B_X$.

Here, $\|x\| = \sqrt{\langle x, x \rangle}$. This definition is an obvious result of the Ω-space which given in [1].

We recall that any normed linear space cannot be an Ω-space. Indeed, if $\|x\| \le \|B_X\|$, then $x = B_X$. Also, $\left\| x/2 \right\| \le \|B_X\|$ implies $x/2 = B_X$. This is not true. So, the concept of Ω-space is meaningless in the normed linear spaces although it is significant for (nonlinear) the quasilinear space.

Example 24: *Let E be an inner product space. Then $\Omega(E)$ and $\Omega_C(E)$ are the inner product Ω-spaces with inner product defined by*

$$\langle A, B \rangle = \{\langle a, b \rangle_E : a \in A, b \in B\}.$$

We only show that $\Omega(E)$ is an inner product Ω-space. Let B_E be a unit sphere of E. Then $B_E \in \Omega(E)$ and $\|B_E\| = 1$. We will show that if $\|A\| \le 1$ then $A \subseteq B_E$. Let x be an arbitrary element of A. Since

$$\|A\| = \sup_{x \in A} \|x\| \le 1$$

we have $\|x\| \le 1$for every $x \in A$. Thus $x \in B_E$.

If $(X, \|\cdot\|)$ is a normed quasilinear space then we know that the relation h given by

$$h_X(x, y) = \inf\{r \ge 0 : x \le y + a_1^r, y \le x + a_2^r \, and \left\|a_i^r\right\| \le r, i = 1,2\}$$

defines a metric on X [1]. Note that, here, $h_X(x, y) = \|x - y\|_X$ may not be satisfied for every $x, y \in X$. But, the inequality

$$h_X(x, y) \le \|x - y\|_X \qquad (2)$$

is always true. This metric is called Hausdorff metric. Because of this inequality, instead of analyzing topological properties of normed quasilinear spaces, analyzing according to the metric derived from this norm is more convenient. Because

$$d(x, y) = \|x - y\|_X \qquad (3)$$

does not define a metric. Therefore, the metric of this norm is not given with the equality of (3). Instead of that, the inequality (2) is the norm metric. If X is a normed linear space, then we know that $h(x, y) = d(x, y)$. So, if a normed quasilinear space is complete according to the norm metric $h(x,y)$ then normed quasilinear space is called complete normed space.

Definition 25: *[9] An inner product quasilinear space is called Hilbert quasilinear space, if it is complete according to the inner-product (norm) metric.* $\Omega_C(R)$ *is a Hilbert quasilinear space.*

Definition 26: *An inner product quasilinear space* X *is called Hilbert Ω-space, if* X *is a Hilbert quasilinear space and Ω-space.*

Example 27: *Let* E *be an inner product space. Then we know that* $\Omega(E)$ *is an inner product quasilinear space and this space is complete with respect to Hausdorff metric; further it is Ω-space [1]. So,* $\Omega(E)$ *is a Hilbert Ω-space.*

Definition 27: *[9] (Orthogonality) The element* x *of an inner product quasilinear space* X *is said to be orthogonal to the an element* $y \in X$ *if*

$$\left\|\langle x,y\rangle\right\|_{\Omega(R)} = 0 .$$

We say also that x and y are orthogonal and write $x \perp y$. Similarly, for subsets $A,B \subseteq X$ we write $x \perp A$ if $x \perp z$ for all $z \in A$ and $A \perp B$ if $a \perp b$ for all $a \in A$ and $b \in B$.

Example 28: *Let us consider the elements* $X = [0,1], Y = \{0\}$ *and* $Z = [-1,0]$ *of* $\Omega(R)$. *Since*

$$\left\|\langle X,Y\rangle\right\|_{\Omega(R)} = \left\|\{a \cdot b : a \in [0,1], b \in \{0\}\}\right\|_{\Omega(R)} = 0$$

We say X and Y orthogonal elements of $\Omega(R)$. Also, since

$$\left\|\langle X,Z\rangle\right\|_{\Omega(R)} = \left\|\{a \cdot b : a \in [0,1], b \in [-1,0]\}\right\|_{\Omega(R)} = 1$$

We have X and Z are not orthogonal elements of $\Omega(R)$.

Remark 29: For every $A,B \in \Omega(R)$, $\langle A,B\rangle = \{a \cdot b : a \in A, b \in B\} = \{0\}$ if and only if at least of A or B equal to $\{0\}$.

An orthonormal subset M of X is an orthogonal set in X consisting elements have norm 1, that is, for all $x,y \in M$

$$\left\|\langle x,y\rangle\right\|_{\Omega(R)} = \begin{cases} 0, x \neq y \\ 1, x = y \end{cases} .$$

Example 30: *Let*

$$A_1 = \{(0,t) : 0 \le t \le 1\}, \quad A_2 = \{(t,0) : 0 \le t \le 1\},$$
$$A_3 = \{(0,-t) : 0 \le t \le 1\}, \quad A_4 = \{(-t,0) : 0 \le t \le 1\}$$

be subsets of $\Omega(R^2)$. From Definition of orthonormality $\{A_1, A_2\}$ and $\{A_3, A_4\}$ are orthonormal.

Theorem 31: *Let* X *be an inner product quasilinear space.* $x \perp y$ *if and only if* $F_x^{\hat{X}} \perp F_y^{\hat{X}}$ *for every* $x, y \in X$.

Proof: Suppose x and y are two orthogonal elements of inner product quasilinear space X . By (24), we have

$$\left\| \langle x, y \rangle \right\|_{\Omega(R)} = \sup \left\{ \left\| \langle a, b \rangle \right\|_{\Omega(R)} : a \in F_x^{\hat{X}}, b \in F_y^{\hat{X}} \right\} = 0.$$

This can only happen if

$$\left\| \langle a, b \rangle \right\|_{\Omega(R)} = 0$$

for every $a \in F_x^{\hat{X}}, b \in F_y^{\hat{X}}$. Therefore $F_x^{\hat{X}} \perp F_y^{\hat{X}}$.

Conversely, if $F_x^{\hat{X}} \perp F_y^{\hat{X}}$ then $\left\| \langle a, b \rangle \right\|_{\Omega(R)} = 0$ for every $a \in F_x^{\hat{X}}, b \in F_y^{\hat{X}}$. In view of the (24), we have $\left\| \langle x, y \rangle \right\|_{\Omega(R)} = 0$. This proves the theorem.

Corollary 32: *Two elements are orthogonal in an inner product quasilinear space* X *if and only if the sets of floors in* X *of these elements are orthogonal in* X_r .

Let X *be an inner product quasilinear space. If*

$$\begin{aligned} \left\| \langle x, y \rangle \right\|_{\Omega(R)} &= \sup \left\{ \left\| \langle a, b \rangle \right\|_{\Omega(R)} : a \in F_x^{\hat{X}}, b \in F_y^{\hat{X}} \right\} \\ &= \sup \left\{ \left\| \langle x, y \rangle \right\|_{\Omega(R)} : b \in F_y^{\hat{X}} \right\} \\ &= 0 \end{aligned}$$

for $x \in X_r$ and $y \in X_s$, then $\langle x, b \rangle = 0$ for every $b \in F_y^{\hat{X}}$.

Theorem 33: *If* $x_1, x_2, \ldots, x_n$ *orthogonal vectors in an inner product quasilinear space, then*

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 \le \sum_{k=1}^{n} \left\| x_k \right\|^2 .$$

The proof of this theorem is analogous to linear counterpart.

Definition 34: *Let* A *be a nonempty subset of an inner product quasilinear space* X. *An element* $x \in X$ *is said to be orthogonal to* A, *denoted by* $x \perp A$, *if* $\|\langle x, y\rangle\|_{\Omega(R)} = 0$ *for every* $y \in A$. *The set of all elements of* X *orthogonal to* A *which is denoted by* $A^{\perp}$ *is called the orthogonal complement of* A *and is indicated by*

$$A^{\perp} = \{x \in X : \|\langle x, y\rangle\|_{\Omega(R)} = 0, y \in A\}$$

For any subset A of an inner product quasilinear space X, $A^{\perp}$ is a closed subspace of X [9].

4. Conclusion

In this study, we prove that the norm of every inner product quasilinear space may not satisfy the parallelogram law. Also introduce the concept of inner product Ω-space and Hilbert Ω-space. Some immediate theorems are also proved. In this paper we aim to give a contribution to the studies on inner product quasilinear spaces by introducing the new results.

Acknowledgements

The authors would like to express their gratitude to the anonymous referees for their helpful comments and suggestions.

References

[1] S.M. Aseev, Quasilinear operators and their application in the theory of multivalued mappings, Proc. Steklov Inst. Math., 2(1986), 23-52.
[2] G. Alefeld and G. Mayer, Interval analysis: Theory and applications, J. Comput. Appl. Math., 121(2000), 421-464.
[3] V. Lakshmikantham, T.G. Bhaskar and J.V. Devi, Theory of Set Differential Equations in Metric Spaces, Cambridge Scientific Publishers, Cambridge, 2006.
[4] M.A. Rojas-Medar, M.D. Jiménez-Gamerob, Y. Chalco-Canoa and A.J. Viera-Brandão, Fuzzy quasilinear spaces and applications, Fuzzy Sets and Systems, 152(2005), 173-190.
[5] S. Çakan, Some new results related to theory of normed quasilinear spaces, Ph.D. Thesis, İnönü University, (2016).
[6] Y. Yılmaz, S. Çakan and Ş. Aytekin, Topological quasilinear spaces, Abstr. Appl. Anal., Article ID 951374(2012), 10 pages.
[7] H.K. Banazılı, On quasilinear operators between quasilinear spaces, MSc. Thesis, İnönü University, (2014).
[8] S. Çakan and Y. Yılmaz, Normed proper quasilinear spaces, J. Nonlinear Sci. Appl., 8(2015), 816-836.
[9] Y. Yılmaz, H. Bozkurt and S. Çakan, On orthonormal sets in inner product quasilinear spaces, Creat. Math. and Inform., 25(2) (2016), 237-247.
[10] H. Bozkurt and Y. Yılmaz, New inner product quasilinear spaces on interval numbers, Journal of Func. Spaces, Article ID 2619271(2016), 9 pages.
[11] H. Bozkurt and Y. Yılmaz, Some new results on inner product qusilinear spaces, Cogent Mathematics, Article ID 1194801(3) (2016), 10 pages.