Information **S**ciences and **C**omputing
Volume 2014, Number 1, Article ID ISC050713, 09 pages
Available online at http://www.infoscicomp.com/

Research Article

Fast Iterative Method (FIM) for Solving Fully Fuzzy Linear Systems

E. Abdolmaleki[a] and S.A. Edalatpanah[a]

[a]Department of Applied Mathematics, Tonekabon Branch, Islamic Azad University, Tonekabon, Iran

Corresponding author: S.A. Edalatpanah; E-mail: saedalat@yahoo.com

Received 07 June 2013; Accepted 12 October 2013

Abstract: In this paper, a new iterative method is applied to find solution of the fully fuzzy linear systems. Furthermore, we show that in some situations that the existing methods such as Jacobi, Gauss-Seidel and SOR are divergent, our proposed method is applicable. Finally, numerical computations are presented based on a particular linear system, which clearly show the reliability and efficiency of our algorithms.

Keywords: Iterative methods; fast iterative methods; fully fuzzy; fuzzy numbers; fuzzy arithmetic; fuzzy linear equations.

1. Introduction

Unfailing real world problems in economics, finance, mechanics etc. can lead to solving a system of linear equations. There are many methods for solving linear systems, see [1-12] and the references therein. Let us consider the following linear systems

$$Ax=b, \qquad (1)$$

However, when the estimation of the system coefficients is imprecise and only some vague knowledge about the actual values of the parameters is available, it may be convenient to represent some or all of them with fuzzy numbers [13]. Fuzzy data is being used as a natural way to describe uncertain data. Fuzzy concept was introduced by Zadeh [13- 14]. We refer the reader to [15] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems including fuzzy metric spaces [16], fuzzy differential equations [17], particle physics [18- 19], Game theory [20], optimization [21] and fuzzy linear systems [22-25].

Friedman *et al*. [22] introduced a general model for solving a fuzzy n × n linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy n × n linear system by a crisp 2n × 2n linear system and studied duality in fuzzy linear systems $AX = BX+Y$ where A and B are real n×n matrices, the unknown vector X is vector consisting of n fuzzy numbers and the constant Y is vector consisting of n fuzzy numbers, in [26]. There are many other numerical methods for solving fuzzy linear systems such as Jacobi, Gauss-Seidel, Adomiam decomposition method and SOR iterative method [27-32]. In addition, other important kind of fuzzy linear systems are the *fully fuzzy linear systems* (FFLS) in which all the parameters are fuzzy numbers. Dehghan and Hashemi [33-34] proposed the Adomian decomposition method, and other iterative methods to find the positive fuzzy vector solution of n × n fully fuzzy linear system. Dehghan *et al.* [35] proposed some computational methods such as Cramer's rule, Gauss elimination method, LU decomposition method and linear programming approach for finding the approximated solution of FFLS. Nasseri *et al.* [36] used a certain decomposition methods of the coefficient matrix for solving fully fuzzy linear system of equations. Kumar *et al.* in [37] obtained exact solution of fully fuzzy linear system by solving a linear programming. In this paper, we use *fast iterative method* (FIM)[38-39] for solving fully fuzzy linear systems.

This paper is organized as follows:

In Section 2 some basic definitions and arithmetic are reviewed. In Section 3 a new method is proposed for solving FFLS and we respectively give the fast iterative method and some convenient iterative methods. In section 4 numerical results are considered to show the efficiency of the proposed method. Section 5 ends this paper with a conclusion.

2. Some Basic Definition and Arithmetic Operations

In this section, an appropriate brief introduction to preliminary topics such as fuzzy numbers and fuzzy calculus will be introduced and the definition for FFLS will be provided. For details, we refer to [33- 37].

Definition 2.1 Let X denote a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function $\mu_{\tilde{A}} : X \to [0,1]$; which assigns a real number $\mu_{\tilde{A}}(x)$ in the interval [0, 1], to each element $x \in X$, where the value of $\mu_{\tilde{A}}(x)$ at x shows the grade of membership of x in $\tilde{A}$.

A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\tilde{A}}(x)$ and is often written $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}$.The class of fuzzy sets on X is denoted with $\Gamma(X)$.

Definition 2.2 A fuzzy set with the following membership function is named a triangular fuzzy number and in this paper we will use these fuzzy numbers.

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{m-x}{\alpha}, & m-\alpha \le x \le m, \alpha > 0, \\ 1 - \dfrac{x-m}{\beta}, & m \le x \le m+\beta, \beta > 0, \\ 0, & else. \end{cases}$$

Definition 2.3 A fuzzy number $\tilde{A}$ is said to be positive (negative) by $\tilde{A} > 0 (\tilde{A} < 0)$ if its membership function $\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x) = 0, \forall x \leq 0 (\forall x \geq 0)$.
Using its mean value and left and right spreads, and shape functions, such a fuzzy number $\tilde{A}$ is symbolically written $\tilde{A} = (m, \alpha, \beta)$. Obviously, $\tilde{A}$ is positive, if and only if $m - \alpha \geq 0$.

Definition 2.4 Two fuzzy numbers $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{B} = (n, \gamma, \delta)$ are said to be equal, if and only if $m = n$, $\alpha = \gamma$ and $\beta = \delta$.

Definition 2.5 Let $\tilde{A} = (m, \alpha, \beta)$, $\tilde{B} = (n, \gamma, \delta)$ be two triangular fuzzy numbers then;

(i) $\tilde{A} \oplus \tilde{B} = (m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta)$,

(ii) **)** $-\tilde{A} = -(m, \alpha, \beta) = (-m, \beta, \alpha)$,

(iii) if $\tilde{A}$ and $\tilde{B}$ be a positive fuzzy numbers then:

$$(m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, n\alpha + m\gamma, n\beta + m\delta),$$

(iv) For scalar multiplication we have;

$$\lambda \otimes (m, \alpha, \beta) = \begin{cases} (\lambda m, \lambda\alpha, \lambda\beta), & \lambda \geq 0, \\ (\lambda m, -\lambda\beta, -\lambda\alpha), & \lambda < 0. \end{cases}$$

Definition 2.6 A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number. A fuzzy matrix $\tilde{A}$ will be positive and denoted by $\tilde{A} > 0$, if each element of $\tilde{A}$ be positive. We may represent n × n fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$, such that $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})$, with the new notation $\tilde{A} = (A, M, N)$, where $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are three n × n crisp matrices.

3. Fast Iterative Method for FFLS

Consider Fully fuzzy linear system (FFLS) $\tilde{A} \otimes \tilde{x} = \tilde{b}$. In this paper we are going to obtain a positive solution of FFLS, where, $\tilde{A} = (A, M, N) > \tilde{0}, \tilde{b} = (b, g, h) > \tilde{0}$ and $\tilde{x} = (x, y, z) > \tilde{0}$.

So we have;

$$(A, M, N) \otimes (x, y, z) = (b, g, h). \tag{2}$$

Then by Definition 2.5 we have;

$$(Ax, Ay + Mx, Az + Nx) = (b, g, h). \tag{3}$$

And by Definition 2.4, concludes that;

$$\begin{cases} Ax = b, \\ Ay + Mx = g, \\ Az + Nx = h. \end{cases} \tag{4}$$

Then,

$$\underbrace{\begin{pmatrix} A & 0 & 0 \\ M & A & 0 \\ N & 0 & A \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b \\ g \\ h \end{pmatrix}}_{\Xi}. \tag{5}$$

So, by assuming that A be a nonsingular matrix we have;

$$\begin{cases} x = A^{-1}b, \\ y = A^{-1}(g - Mx), \\ z = A^{-1}(h - Nx). \end{cases} \tag{6}$$

Dehghan *et al.* [33] applied some iterative techniques such as Richardson, Jacobi, Jacobi overrelaxation(JOR), Gauss–Seidel, successive overrelaxation (SOR), accelerated overrelaxation (AOR), symmetric and unsymmetric SOR (SSOR and USSOR) and extrapolated modified Aitken (EMA) for solving FFLS. First, we review their work.

Consider Eq. (4) and let $A=Q-P$ be a proper splitting of crisp matrix A and Q, called the splitting matrix, be a nonsingular crisp matrix. Thus, the iterative method for FFLS is as follows;

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = T \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} + \xi, (k \geq 0). \tag{7}$$

T is called the iteration matrix and ξ is a vector and;

$$T = \begin{pmatrix} Q^{-1}P & 0 & 0 \\ -Q^{-1}M & Q^{-1}P & 0 \\ -Q^{-1}N & 0 & Q^{-1}P \end{pmatrix}, \xi = \begin{pmatrix} Q^{-1}b \\ Q^{-1}g \\ Q^{-1}h \end{pmatrix}. \tag{8}$$

Therefore by choose special parameters in Q we can obtain the popular iterative methods. For example, if $A=D-L-U$, where D is diagonal, L is lower triangular and U is upper triangular part of A, then we have;

1) Jacobi method for $Q=D$.
2) JOR (Jacobi Overrelaxation) method for $Q = \frac{1}{w}D, (w \in R)$.
3) Gauss-Seidel method for $Q=D-L$.

4) SOR method for $Q=(\frac{1}{w}D-L),(w\in R)$.

For details, we refer to [33].

Next, we apply another acceleration method called *fast iterative method* (FIM) for FFLS.

Saberi Najafi and Edalatpanah in [38] proposed some iterative methods called FIM for solving linear systems. They presented that these new algorithms are faster than AOR, SOR, SSOR, etc. Besides, they have shown that a divergent process can be converges by using FIM methods.

Consider Eq. (5) and Let $\Lambda = \Sigma - \Gamma - \Phi$,
where,

$$\Sigma=\begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}, \Gamma=\begin{pmatrix} L & 0 & 0 \\ -M & L & 0 \\ -N & 0 & L \end{pmatrix}, \Phi=\begin{pmatrix} U & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U \end{pmatrix}$$

and,

$$A=D-L-U.$$

Based on above demonstration, we describe FIM in the following algorithms;

<u>Two- steps FIM</u>

$$\begin{cases} X^{(i+\frac{1}{2})}=\Sigma^{-1}[\Xi+\Gamma X^{(i+\frac{1}{2})}+\Phi X^{(i)}], \\ X^{(i+1)}=\Sigma^{-1}[\Xi+\Gamma X^{(i+1)}+\Phi\{(1-w)X^{(i+\frac{1}{2})}+wX^{(i)}\}] \quad ,w\in R. \end{cases}$$

In general, for $n \geq 2$ we have;

<u>n- steps FIM</u>

$$\begin{cases} X^{(i+\frac{1}{n})}=\Sigma^{-1}[\Xi+\Gamma X^{(i+\frac{1}{n})}+\Phi X^{(i)}], \\ X^{(i+\frac{2}{n})}=\Sigma^{-1}[\Xi+\Gamma X^{(i+\frac{2}{n})}+\Phi X^{(i+\frac{1}{n})}], \\ \vdots \\ X^{(i+\frac{n-1}{n})}=\Sigma^{-1}[\Xi+\Gamma X^{(i+\frac{n-1}{n})}+\Phi X^{(i+\frac{n-2}{n})}], \\ X^{(i+1)}=\Sigma^{-1}[\Xi+\Gamma X^{(i+1)}+\Phi\{(1-w)X^{(i+\frac{n-1}{n})}+wX^{(i+\frac{n-2}{n})}\} \quad ,w\in R. \end{cases}$$

Theorem 3.1 FIM for solving fully fuzzy linear system $\tilde{A}\otimes\tilde{X}=\tilde{b}$, converges if and only if its classical version converges for solving the crisp linear system $Ax = b$ derived from the corresponding FFLS.

Proof. By above demonstrations and based on Eq. (8) it is easy to see that spectrum of T is equal to spectrum of $Q^{-1}P$. Therefore, the proof is complete.

4. Numerical Experiments

In this section, we give some numerical experiments to illustrate the results obtained in previous sections. All the numerical experiments presented in this section were computed in double precision using a MATLAB 7 on a PC with a 1.86GHz 32-bit processor and 1GB memory.

Example 4.1 Consider the Consider the following FFLS:

$$\tilde{A} = (A, M, N); \begin{cases} A = tridiag(1,4,1)_{m\times m}, \\ M = tridiag(0.1,1,0.1)_{m\times m}, \\ N = tridiag(0.2,1,0.1)_{m\times m}. \end{cases}$$

And,

$$\tilde{b} = (b, g, h); b_i = i, g_i = \frac{i}{m}, h_i = \frac{i}{m+1}.$$

The following tables show the numerical results of above example with the tolerance $\varepsilon = 10^{-6}$ and the initial approximation zero vector. In the Table 1, we reported the number of iterations (**Iter**) and Elapsed time (**ELP**) for the *Jacobi*, *Gauss-Seidel* and *SOR* iterative method with different *m* and *w*.

Table 1: Shows the results of example 4.1 by Jacobi, Gauss-Seidel and SOR methods

Method	**Jacobi method**		**Gauss-Seidel method**		**SOR method**		
m	**Iter**	**ELP**	**Iter**	**ELP**	***w***	**Iter**	**ELP**
50	33	0.008131	16	0.003316	1.05	15	0.002749
100	34	0.017908	17	0.009510	1.01	16	0.008054
200	36	0.095731	17	0.061495	1.03	16	0.025836
300	37	0.122997	18	0.070193	1.02	17	0.068845
400	38	0.248280	18	0.108688	1.02	17	0.103024
500	38	0.476124	18	0.180213	1.02	17	0.199463
600	39	0.694309	18	0.462218	1.02	18	0.315574
700	39	0.751305	18	0.534026	1.02	18	0.525878

Table 2: Shows the results of example 4.1 by Two-steps FIM iterative method

Method	**Two step FIM**		
m	***w***	**Iter**	**ELP**
50	-0.1	8	0.001137
100	-0.1	8	0.003427
200	-0.1	8	0.009236
300	-0.1	8	0.021059
400	-0.1	8	0.035997
500	-0.1	9	0.090477
600	-0.1	9	0.142446
700	-0.1	9	0.200639

In the Table 2, we reported the number of iterations (**Iter**) and Elapsed time (**ELP**) for the *Two-steps FIM* iterative method with different *m* and *w*.

Example 4.2 Consider the following FFLS:

$$\begin{bmatrix} (1,0.2,0.2) & (1,0.4,0.3) & (2,0.3,0.4) & (4,0.2,0.1) \\ (4,0.3,0.1) & (3,0.4,0.2) & (2,0.2,0.3) & (1,0.1,0.3) \\ (1,0.3,0.2) & (1,0.5,0.2) & (3,0.3,0.1) & (1,0.2,0.3) \\ (2,0.4,0.5) & (4,0.5,0.2) & (2,0.6,1.2) & (3,0.3,0.3) \end{bmatrix} \begin{pmatrix} (x_1,y_1,z_1) \\ (x_2,y_2,z_2) \\ (x_3,y_3,z_3) \\ (x_4,y_4,z_4) \end{pmatrix} = \begin{pmatrix} (6.9,5,4.1) \\ (5,3,4) \\ (4.9,4,3.2) \\ (7,5,6) \end{pmatrix}.$$

If we use iterative methods [33] for this problem we can see that all of the Jacobi, Gauss-Sidel and JOR, SOR methods are divergent (since $\rho(Q^{-1}P) > 1$).

However, by using the initial values $x = y = z = s = (0,0,0,0)^t$ and stopping criterion $tol \le 10^{-6}$, the Two-steps FIM method ($w = 0.7$) converges in 11 iterations to the following exact solution;

$$\begin{pmatrix} (x_1,y_1,z_1) \\ (x_2,y_2,z_2) \\ (x_3,y_3,z_3) \\ (x_4,y_4,z_4) \end{pmatrix} = \begin{pmatrix} (0.1961,0.0072,0.1448) \\ (0.3143,0.0393,0.3681) \\ (1.1169,0.8533,0.5744) \\ (1.0390,0.6348,0.4386) \end{pmatrix}.$$

5. Conclusions

In this paper, the fully fuzzy linear systems, *i.e.*, fuzzy linear systems with fuzzy coefficients involving fuzzy variables are investigated and fast iterative method (FIM) is applied for solving these systems. The proposed method is easy to understand and apply in real life situations. Also, we show that our algorithm compare with some other algorithms works better. Finally, from theoretical speaking and numerical examples, it may be concluded that this method is efficient and convenient.

Acknowledgements

The authors would like to thank Tonekabon Branch of Islamic Azad University of Iran for the financial support of this research, which is based on a research project contract.

References

[1] R.S. Varga, Matrix Iterative Analysis (Second ed.), Springer, Berlin, 2000.
[2] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.
[3] M.M. Martins, M.E. Trigo and D.J. Evans, An iterative method for positive real systems, Int. J. Comput. Math., 84(2007), 1603-1611.
[4] L.T. Zhang, T.Z. Huang, S.H. Cheng and Y.P. Wang, Convergence of a generalized MSSOR method for augmented systems, J. Comput. Appl. Math., 236(1) (2012), 1841-1850.

[5] H.S. Najafi and S.A. Edalatpanah, Some improvements in PMAOR method for solving linear systems, J. Inform. Comp. Sci., 6(2011), 15-22.
[6] H.S. Najafi and S.A. Edalatpanah, The block AOR iterative methods for solving fuzzy linear systems, The Journal of Mathematics and Computer Science, 4(4) (2012), 527-535.
[7] H.S. Najafi and S.A. Edalatpanad, On application of Liao's method for system of linear equations, Ain Shams Engineering Journal, 4(2013), 501-505.
[8] H.S. Najafi and S.A. Edalatpanah, Comparison analysis for improving preconditioned SOR-type iterative method, Numerical Analysis and Applications, 6(2013), 62-70.
[9] H.S. Najafi and S.A. Edalatpanah, Iterative methods with analytical preconditioning technique to linear complementarity problems: Application to obstacle problems, RAIRO - Operations Research, 47(2013), 59-71.
[10] H.S. Najafi and S.A. Edalatpanah, On the convergence regions of generalized AOR methods for linear complementarity problems, J. Optim. Theory. Appl., 156(2013), 859-866.
[11] H.S. Najafi and S.A. Edalatpanad, Two stage mixed-type splitting iterative methods with applications, J. Taibah Univ. Sci., 7(2013), 35-43.
[12] H.S. Najafi and S.A. Edalatpanah, A new modified SSOR iteration method for solving augmented linear systems, Int. J. Comput. Math., (2013).
[13] L.A. Zadeh, Fuzzy sets, Information and Control, 8(1965), 338-353.
[14] L.A. Zadeh, A fuzzy-set-theoretic interpretation of linguistic hedges, Journal of Cybernetics, 2(1972), 4-34.
[15] A. Kaufmann and M.M. Gupta, Introduction Fuzzy Arithmetic, Van Nostrand Reinhold, New York, 1985.
[16] J.H. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons & Fractals, 22(2004), 1039-1046.
[17] M. Freidman, M. Ming and A. Kandel, Numerical solutions of fuzzy differential and integral equations, Fuzzy Sets and Systems, 106(1999), 35-48.
[18] M.S. Elnaschie, A review of E-infinity theory and the mass spectrum of high energy particle physics, Chaos, Solitons & Fractals, 19(2004), 209-236.
[19] M.S. Elnaschie, The concepts of E infinity: An elementary introduction to the Cantorian fractal theory of quantum physics, Chaos, Solitons & Fractals, 22(2004), 495-511.
[20] H.S. Najafi and S.A. Edalatpanah, On the Nash equilibrium solution of fuzzy bimatrix games, International Journal of Fuzzy Systems and Rough Systems, 5(2) (2012), 93-97.
[21] H.S. Najafi and S.A. Edalatpanah, A note on a new method for solving fully fuzzy linear programming problems, Applied Mathematical Modelling, 37(2013), 7865-7867.
[22] M. Friedman, M. Ming and A. Kandel, Fuzzy linear systems, Fuzzy Sets and Systems, 96(1998), 201-209.
[23] T. Allahviranloo, Successive over relaxation iterative method for fuzzy system of linear equations, Appl. Math. Comput., 162(2005), 189-196.
[24] T. Allahviranloo, The Adomian decomposition method for fuzzy system of linear equations, Appl. Math. Comput., 163(2005), 553-563.
[25] H.S. Najafi and S.A. Edalatpanah, Preconditioning strategy to solve fuzzy linear systems (FLS), International Review of Fuzzy Mathematics, 7(2) (2012), 65-80.
[26] M. Friedman, M. Ming and A. Kandel, Duality in fuzzy linear systems, Fuzzy Sets and Systems, 109(2000), 55-58.
[27] T. Allahviranloo, Numerical methods for fuzzy system of linear equations, Applied Mathematics and Computation, 155(2004), 493-502.
[28] T. Allahviranloo, Successive over relaxation iterative method for fuzzy system of linear equations, Applied Mathematics and Computation, 162(2005), 189-196.
[29] T. Allahviranloo, The Adomian decomposition method for fuzzy system of linear equations, Applied Mathematics and Computation, 163(2005), 553-563.
[30] M. Dehghan and B. Hashemi, Iterative solution of fuzzy linear systems, Applied Mathematics and Computation, 175(2006), 645-674.

[31] S.H. Nasseri and M. Sohrabi, Gram-Schmidt approach for linear system of equations with fuzzy parameters, The Journal of Mathematics and Computer Science, 1(2010), 80-89.
[32] H.S. Najafi and S.A. Edalatpanah, An improved model for iterative algorithms in fuzzy linear systems, Computational Mathematics and Modeling, 24(2013), 443-451.
[33] M. Dehghan, B. Hashemi and M. Ghatee, Solution of the fully fuzzy linear systems using iterative techniques, Chaos Solutions and Fractals, 34(2007), 316-336.
[34] M. Dehghan and B. Hashemi, Solution of the fully fuzzy linear systems using the decomposition procedure, Applied Mathematics and Computation, 182(2006), 1568-1580.
[35] M. Dehghan, B. Hashemi and M. Ghati, Computational methods for solving fully fuzzy linear systems, Appl Math and Comput., 179(2006), 328-343.
[36] S.H. Nasseri, M. Sohrabi and E. Ardil, Solving fully fuzzy linear systems by use of a certain decomposition of the coefficient matrix, International Journal of Computational and Mathematical Sciences, 2(2008), 140-142.
[37] A. Kumar, J. Kaur and P. Singh, A new method for solving fully fuzzy linear programming problems, Appl. Math. Comput., 35(2011), 817-823.
[38] H.S. Najafi and S.A. Edalatpanah, Fast iterative method-FIM: Application to the convection- diffusion equation, J. Inform. Comp. Sci., 6(2011), 303-313.
[39] H.S. Najafi, S.A. Edalatpanah and B.P. Moghaddam, Some approaches for using stationary iterative methods to linear equations generated from the boundary element method, International Journal of Applied Mathematics and Physics, 4(2) (2012), 131-136.
