Information in **S**ciences and **C**omputing
Volume 2015, Article ID ISC460115, 07 pages
Available online at http://www.infoscicomp.com/

Research Article

Inequalities Including Ruscheweyh Derivative Operator

P. Arjomandinia[a] **and S. Shams**[a]

[a]Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran

Corresponding author: P. Arjomandinia, E-mail: p.arjomandinia@urmia.ac.ir

Received 04 January 2015; Accepted 12 February 2015

Abstract: In the present paper we introduce the subclasses $M_\lambda(a, b, \alpha)$ and $S(\alpha, \beta, \lambda)$ of the class of normalized analytic functions $\mathcal{A}$. By using Jack's lemma and its result we deduce some relations between the functions which satisfy in the special differential inequalities in the open unit disk D. Corollaries of the main results are also mentioned.

Keywords: Jack's lemma; Normalized analytic function; Ruscheweyh derivative operator; Starlike function.

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions $f(z)$ in $D = \{z \in C : |z| < 1\}$ which satisfy the normalization condition, $f(0) = f'(0) - 1 = 0$. Such a function $f \in \mathcal{A}$ has the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, \quad (z \in D). \tag{1}$$

A starlike function is a conformal mapping of the unit disk onto a domain starlike with respect to the origin. Also, a function $f \in \mathcal{A}$ is starlike of order α with $0 \leq \alpha < 1$, if and only if $\Re \frac{zf'(z)}{f(z)} > \alpha$ for all $z \in D$. It is well known that a function $f \in \mathcal{A}$ is starlike if and only if $\Re \frac{zf'(z)}{f(z)} > 0$ on D, (see [1]).

Now, let $f, g \in \mathcal{A}$ are given by Teylor series expansions of the forms

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in D).$$

The Hadamard product (or convolution) of f and g, denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{2}$$

Suppose that $f \in \mathcal{A}$. The Ruscheweyh derivative operator [5], $R^\lambda : \mathcal{A} \to \mathcal{A}$ is defined as follows

$$R^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \quad (\lambda > -1, z \in D). \tag{3}$$

It is easy to see that

$$R^0 f(z) = f(z),\ R^1 f(z) = zf'(z),\ R^2 f(z) = \frac{z}{2}(2f'(z) + zf''(z))$$

and so on. Hence, it can be shown that for each $n \in N$

$$R^n f(z) = \frac{z}{n!}\frac{d^n}{dz^n}(z^{n-1}f(z)).$$

Also, using (3) and strightforward calculations we deduce that for each $\lambda > -1$ and $z \in D$

$$z(R^\lambda f)'(z) = (\lambda+1)R^{\lambda+1}f(z) - \lambda R^\lambda f(z). \tag{4}$$

Definition 1.1 *Let $\lambda > -1$ and $f \in \mathcal{A}$. The function $f(z)$ is said to be in the class $M_\lambda(a, b, \alpha)$ if it satisfies the inequality*

$$\left|\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - 1\right|^a \left|\frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} - 1\right|^b \le (1-\alpha)^b \left(\frac{\lambda+1}{\lambda+2}\ \alpha\right)^a \tag{5}$$

with $a \ge 0,\ a + b \ge 0$ and $0 < \alpha < 1$.

In the special case, for $a = \lambda = 0$ and $b = 1$, we obtain the subclass $M_0(0, 1, \alpha) = M(1, \alpha)$ given by

$$M(1,\alpha) = \left\{ f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| \le 1 - \alpha,\ z \in D \right\}.$$

Definition 1.2 *For $\lambda > -1$ and $f \in \mathcal{A}$ we say that the function $f(z)$ belongs to the subclass $S(\alpha, \beta, \lambda)$ if it satisfies the inequality*

$$\Re\left\{ \alpha\frac{z(R^{\lambda+1}f)'(z)}{R^\lambda f(z)} + \beta\frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} \right\} > \beta - \frac{\alpha + 2\beta}{2(\lambda+1)} \tag{6}$$

where, $\alpha > 0$ and $\alpha + \beta \ge 0$.

In the special case if $\beta = \lambda = 0$, then we obtain the subclass $S(\alpha, 0, 0) = S(\alpha)$ given by

$$S(\alpha) = \left\{ f \in \mathcal{A} : \Re\left\{ \frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} > -\frac{1}{2},\ z \in D \right\}.$$

In 2005 Singh, Singh and Gupta [7] proved that if $f(z) \in \mathcal{A}$ satisfies the differential inequality

$$\Re\left((1-\alpha)f'(z) + \alpha(1 + \frac{zf''(z)}{f'(z)}) \right) > 0; \quad (0 < \alpha < 1,\ z \in D)$$

then f is univalent in D.

In 2009 Singh, Gupta and Singh [6] showed that if $f(z) \in \mathcal{A}$ satisfies the differential inequality

$$\Re\left((1-\alpha)f'(z) + \alpha(1 + \frac{zf''(z)}{f'(z)})\right) < \beta; \quad (1 < \beta \leq \alpha,\ z \in D)$$

then $\Re(f'(z)) > 0$ and hence f is univalent.

Motivated by the work of Singh, Gupta and Singh [6], and using the similar techniques applied in [2], in the present paper we aim to obtain some interesting relations between the analytic functions which satisfy in certain inequalities in the open unit disk. In order to prove our main results we shall use the following lemmas. The first lemma was stated in 1971 and is known as Jack's lemma.

Lemma 1.3 ([3]) *Let $z_0 \in D, r_0 = |z_0|$ and $D_{r_0} = \{z \in C : |z| < r_0\}$. Suppose that*

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots, \quad (n \in N)$$

is continuos in $\overline{D}_{r_0}$ and analytic in $D_{r_0} \cup \{z_0\}$ with $f(z) \not\equiv 0$. If

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{D}_{r_0}\},$$

then there exists an $m \geq n$ such that $z_0 f'(z_0) = mf(z_0)$.

Lemma 1.4 ([4]) *Suppose that the function $J : C^2 \to C$ satisfies the condition*

$$\Re(J(is, t)) \leq 0, \quad \left(s \in R, t \leq \frac{-n(1+s^2)}{2}, n \in N\right).$$

If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots,$ is analytic in D and

$$\Re(J(p(z), zp'(z))) > 0, \quad (z \in D)$$

then $\Re(p(z)) > 0$ for all $z \in D$.

2 Main Results

In the first theorem, we determine conditions on the parameters α, λ so that $M_\lambda(a, b, \alpha) \subseteq K_\lambda(\alpha)$. To prove this, we define a function $w(z)$ and use Jack's lemma.

Theorem 2.1 *If $\lambda > -1, 0 \leq \alpha \leq \frac{1}{2}$ and $f \in M_\lambda(a, b, \alpha)$, then $f \in K_\lambda(\alpha)$, where*

$$K_\lambda(\alpha) = \left\{f \in \mathcal{A} : \Re\left(\frac{R^{\lambda+1}f(z)}{R^\lambda f(z)}\right) > \alpha,\ z \in D\right\}. \tag{7}$$

Proof: For the function $f(z) = z + a_2z^2 + \ldots \in M_\lambda(a, b, \alpha)$, consider the analytic function $p(z)$ by

$$p(z) = \frac{R^{\lambda+1}f(z)}{R^\lambda f(z)}, \ (z \in D).$$

We have to show that $\Re(p(z)) > \alpha$. To prove this, we define the analytic function $w(z)$ on the open set $G = D \setminus \{z : p(z) = 2\alpha - 1\}$ by

$$w(z) = \frac{p(z) - 1}{p(z) - (2\alpha - 1)}.$$

The definition of $p(z)$ shows that $p(0) = 1$. Hence, we obtain $w(0) = 0$. Also, it is easy to see that $|w(z)| < 1$ if and only if $\Re(p(z)) > \alpha$. So, it is sufficient to show that $|w(z)| < 1$. Suppose, (for a contradiction) that there is a point $z_0 \in G$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1, \quad (z \in G).$$

By Lemma 1.3 there exists an $m \geq 1$ such that $z_0w'(z_0) = mw(z_0)$. We have

$$\frac{R^{\lambda+1}f(z)}{R^\lambda f(z)} - 1 = p(z) - 1 = \frac{2(1-\alpha)w(z)}{1 - w(z)}, \quad (z \in G).$$

In addition, by using (4), we obtain

$$\begin{aligned} \frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - 1 &= \frac{1}{\lambda+2}\left(\frac{z(R^{\lambda+1}f)'(z)}{R^{\lambda+1}f(z)} - 1\right) \\ &= \frac{1}{\lambda+2}\left(\frac{zp'(z)}{p(z)} + \frac{z(R^\lambda f)'(z)}{R^\lambda f(z)} - 1\right) \\ &= \frac{1}{\lambda+2}\left(\frac{zp'(z)}{p(z)} + (\lambda+1)(p(z) - 1)\right). \end{aligned} \tag{8}$$

By taking

$$p(z) = \frac{1 - (2\alpha - 1)w(z)}{1 - w(z)}$$

in place of $p(z)$ in (8), we have

$$\begin{aligned} &\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - 1 \\ &= \frac{1}{\lambda+2}\left\{\frac{zw'(z)}{1-w(z)} + \frac{(1-2\alpha)zw'(z)}{1+(1-2\alpha)w(z)} + (\lambda+1)\left(\frac{1-(2\alpha-1)w(z)}{1-w(z)} - 1\right)\right\} \\ &= \frac{1}{\lambda+2}\left\{\frac{zw'(z) + 2(\lambda+1)(1-\alpha)w(z)}{1-w(z)} + \frac{(1-2\alpha)zw'(z)}{1+(1-2\alpha)w(z)}\right\}. \end{aligned}$$

Now, by putting $z = z_0$ in the last equation we find that

$$\frac{R^{\lambda+2}f(z_0)}{R^{\lambda+1}f(z_0)} - 1 = \frac{1}{\lambda+2}\left\{\frac{mw(z_0) + 2(\lambda+1)(1-\alpha)w(z_0)}{1-w(z_0)} + \frac{(1-2\alpha)mw(z_0)}{1+(1-2\alpha)w(z_0)}\right\}$$

$$= \frac{w(z_0)}{\lambda+2}\left\{\frac{(1+(1-2\alpha)w(z_0))(m+2(\lambda+1)(1-\alpha))+m(1-w(z_0))(1-2\alpha)}{(1-w(z_0))(1+(1-2\alpha)w(z_0))}\right\}$$

$$= \frac{2(1-\alpha)w(z_0)}{\lambda+2}\left\{\frac{m+\lambda+1+(\lambda+1)(1-2\alpha)w(z_0)}{(1-w(z_0))(1+(1-2\alpha)w(z_0))}\right\}.$$

Using the fact that $|w(z_0)| = 1$ and $m \geq 1$, from the last equation we conclude that

$$\begin{aligned}
\left|\frac{R^{\lambda+2}f(z_0)}{R^{\lambda+1}f(z_0)}-1\right|^a \left|\frac{R^{\lambda+1}f(z_0)}{R^{\lambda}f(z_0)}-1\right|^b &= \frac{|m+\lambda+1+(\lambda+1)(1-2\alpha)w(z_0)|^a}{|1-w(z_0)|^{a+b}|1+(1-2\alpha)w(z_0)|^a} \\
&\times \frac{2^{a+b}(1-\alpha)^{a+b}}{(\lambda+2)^a} \\
&\geq \frac{(m+\lambda+1-(\lambda+1)(1-2\alpha))^a 2^b(1-\alpha)^{a+b}}{(\lambda+2)^a 2^{a+b}(1-\alpha)^a} \\
&\geq (1-\alpha)^b\left(\frac{1+2\alpha(\lambda+1)}{2(\lambda+2)}\right)^a \\
&> (1-\alpha)^b\left(\frac{\lambda+1}{\lambda+2}\alpha\right)^a,
\end{aligned}$$

which is a contradiction to the fact that $f(z) \in M_\lambda(a,b,\alpha)$. So, we should have $|w(z)| < 1$ or equivalently $\Re(p(z)) > \alpha$. This completes the proof.

Corollary 2.2 *If the function $f \in \mathcal{A}$ satisfies in one of the following conditions:*
i) $\left|1-\frac{zf'(z)}{f(z)}\right| \leq \frac{1}{2}$, *or*
ii) $\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{2}$
for all $z \in D$, then f is starlike of order $\frac{1}{2}$.

Proof: This is a direct consequence of Theorem 2.1 by taking $\lambda = a = 0, \alpha = 1/2$ and $b = 1$ for (i), and $\lambda = b = 0, \alpha = 1/2$ and $a = 1$ for (ii).

In the next theorem we use another technique to prove the desired result.

Theorem 2.3 *Let $\lambda > -1, \alpha > 0$ and $\alpha+\beta \geq 0$. If $f \in S(\alpha,\beta,\lambda)$, then $f \in K_\lambda(\frac{\lambda}{\lambda+1})$, where $K_\lambda(\frac{\lambda}{\lambda+1})$ is given by (7).*

Proof: For the function $f(z) \in S(\alpha,\beta,\lambda)$, we define the analytic function $p(z)$ on D by $p(z) = (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} - \lambda$. It is clear that $p(0) = 1$. The identity (4) after an easy computation shows that

$$\begin{aligned}
\alpha\frac{z(R^{\lambda+1}f)'(z)}{R^{\lambda}f(z)} + \beta\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} &= \frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\left\{\frac{\alpha z(R^{\lambda+1}f)'(z)}{R^{\lambda+1}f(z)}+\beta\right\} \\
&= \left(\frac{p(z)+\lambda}{\lambda+1}\right)\left\{\alpha\left(\frac{zp'(z)}{p(z)+\lambda}+p(z)\right)+\beta\right\} \\
&= \frac{\alpha zp'(z)}{\lambda+1} + \left(\frac{p(z)+\lambda}{\lambda+1}\right)(\alpha p(z)+\beta).
\end{aligned}$$

Hence, $f(z) \in S(\alpha, \beta, \lambda)$ if and only if

$$\Re\{\alpha z p'(z) + (p(z) + \lambda)(\alpha p(z) + \beta) + \frac{\alpha}{2} - \beta\lambda\} > 0.$$

We define the function $J : C^2 \to C$ by

$$J(u, v) = \alpha v + (u + \lambda)(\alpha u + \beta) + \frac{\alpha}{2} - \beta\lambda.$$

By the assumption of the theorem, we have $\Re(J(p(z), zp'(z))) > 0$. Let $u_1, v_1 \in R$ and $v_1 \leq \frac{-(1+u_1^2)}{2}$. We show that $\Re(J(iu_1, v_1)) \leq 0$. To prove this, we write

$$\begin{aligned} J(iu_1, v_1) &= \alpha v_1 + (iu_1 + \lambda)(i\alpha u_1 + \beta) + \frac{\alpha}{2} - \beta\lambda \\ &= \alpha v_1 + \frac{\alpha}{2} - \alpha u_1^2 + iu_1(\beta + \lambda\alpha). \end{aligned}$$

So, we obtain

$$\begin{aligned} \Re(J(iu_1, v_1)) &= \alpha v_1 + \frac{\alpha}{2} - \alpha u_1^2 \\ &\leq -\frac{\alpha}{2}(1 + u_1^2) + \frac{\alpha}{2} - \alpha u_1^2 \\ &= -\frac{3}{2}\alpha u_1^2 \leq 0. \end{aligned}$$

Therefore, we conclude that all conditions of Lemma 1.4 are satisfied and we have $\Re(p(z)) > 0$, or equivalently $\Re\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right) > \frac{\lambda}{\lambda+1}$ and the proof is complete.

Finally, by taking $\lambda = 0$ and $\beta = -\alpha$ in the above theorem we obtain the following result.

Corollary 2.4 *If $f \in \mathcal{A}$ and $\Re\left(\frac{z^2 f''(z)}{f(z)}\right) > -\frac{1}{2}$, then f is starlike in D.*

Acknowledgements: The authors would like to thank the referees for their helpful comments and suggestions to improve our manuscript.

References

[1] P.L. Duren, Univalent Functions, Springer Verlag, New York, 1983.

[2] A. Ebadian and Sh. Najafzadeh, Inequalities for meromorphically p-valent functions, Iranian Journal of Science and Technology, Transaction A, 33(A2) (2009), 139-143.

[3] I.S. Jack, Functions starlike and convex of order α, J. London Math. Soc., 3(1971), 469-474.

[4] S.S. Miller, Differential inequalities and Caratheodory functions, Bull. Amer. Math. Soc., 81(1975), 79-81.

[5] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., 49(1975), 109-115.

[6] S. Singh, S. Gupta and S. Singh, On a problem in the theory of univalent functions, Gene. Math., 17(3) (2009), 135-139.

[7] V. Singh, S. Singh and S. Gupta, A problem in the theory of univalent functions, Integral Transforms and Special Functions, 16(2) (2005), 179-186.
