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Research Article

Certain Subclass of Analytic Functions Defined By Multiplier Operator

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Abstract: In this paper, the author introduces a new class $\mathcal{S}^*_{\lambda,\mu}(m, s, \alpha; \eta)$ of analytic functions which is defined by means of operators $\mathcal{K}^{m,s}_{\lambda,\mu,\alpha}$ in the open unit disk $\mathbb{U}$. The operator $\mathcal{K}^{m,s}_{\lambda,\mu,\alpha}$ is defined as the iterations of simple form of differential and integral operators. By using well known lemma and identity, the sufficient conditions for an analytic function f to be member of the class $\mathcal{S}^*_{\lambda,\mu}(m, s, \alpha; \eta)$ is obtained. Several corollaries of the main result are also pointed out. The concluding section describe the possibility of further extension of the present work to p-valent functions.

Keywords: Analytic function; Univalent function; Starlike function; Convex function; Differential and integral operator; Linear multiplier operator.

1 Introduction and Definition

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the *open unit disk*

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and satisfies the usual *normalized* conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathbb{U}$.
A function $f \in \mathcal{A}$ is said to be starlike of order η if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \eta \quad (z \in \mathbb{U}) \tag{1.2}$$

for some η $(0 \le \eta < 1)$. We denote by $\mathcal{S}^*(\eta)$, the class of all functions in $\mathcal{A}$ which are starlike of order η in $\mathbb{U}$. Clearly, $\mathcal{S}^*(\eta) \subseteq \mathcal{S}^*(0) = \mathcal{S}^*$ $(0 \le \eta < 1)$. Furthermore, a function $f \in \mathcal{A}$ is said to be convex of order η if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \eta \quad (z \in \mathbb{U}) \tag{1.3}$$

for some η $(0 \le \eta < 1)$. We denote the class of convex functions of order η by $\mathcal{CV}(\eta)$. Clearly, $\mathcal{CV}(\eta) \subseteq \mathcal{CV}(0) = \mathcal{CV}$ (see [3, 11]).

Now we introduce the operator for purpose of defining the class.

For $\mu > -1,\ \lambda \ge 0$, define an integral operator $\mathcal{I}_{\lambda,\mu} : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$\mathcal{I}_{\lambda,\mu} f(z) = \begin{cases} \frac{\mu+1}{\lambda} z^{1-\frac{\mu+1}{\lambda}} \int_0^z t^{\frac{\mu+1}{\lambda}-2} f(t)dt & (\lambda \ne 0), \\ f(z) & (\lambda = 0), \end{cases} \tag{1.4}$$

and a differential operator $\mathcal{D}_{\lambda,\mu} : \mathcal{A} \longrightarrow \mathcal{A}$ given by

$$\mathcal{D}_{\lambda,\mu} f(z) = \begin{cases} \frac{\lambda}{\mu+1} z^{2-\frac{\mu+1}{\lambda}} \frac{d}{dz}(z^{\frac{\mu+1}{\lambda}-1} f(z)) & (\lambda \ne 0) \\ f(z) & (\lambda = 0). \end{cases} \tag{1.5}$$

For $m \in \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\},\ \mu > -1,\ \lambda > 0$, define a linear operator $\mathcal{T}^m_{\lambda,\mu} : \mathcal{A} \longrightarrow \mathcal{A}$ in terms of the operators $\mathcal{I}_{\lambda,\mu}$ and $\mathcal{D}_{\lambda,\mu}$ as follows:

$$\mathcal{T}^m_{\lambda,\mu} f(z) = \begin{cases} \mathcal{I}_{\lambda,\mu} \mathcal{T}^{m+1}_{\lambda,\mu} f(z), & m \in \mathbb{Z}^- \\ \mathcal{D}_{\lambda,\mu} \mathcal{T}^{m-1}_{\lambda,\mu} f(z), & m \in \mathbb{Z}^+ \\ f(z) & m = 0. \end{cases} \tag{1.6}$$

Thus, for a function $f \in \mathcal{A}$ given by (1.1), it follows from (1.6) that

$$\mathcal{T}^m_{\lambda,\mu} f(z) = z + \sum_{n=2}^{\infty} \left[1 + \frac{\lambda(n-1)}{\mu+1}\right]^m a_n z^n \quad (m \in \mathbb{Z},\ \mu > -1,\ \lambda > 0;\ z \in \mathbb{U}). \tag{1.7}$$

Now, in terms of $\mathcal{T}^m_{\lambda,\mu}$, we introduce the linear multiplier operator $\mathcal{K}^{m,s}_{\lambda,\mu,\alpha} : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$\begin{aligned} \mathcal{K}^{m,0}_{\lambda,\mu,\alpha} f(z) &= \mathcal{T}^m_{\lambda,\mu} f(z) \\ \mathcal{K}^{m,1}_{\lambda,\mu,\alpha} f(z) &= (1-\alpha)\mathcal{T}^m_{\lambda,\mu} f(z) + \alpha z(\mathcal{T}^m_{\lambda,\mu} f(z))' = \mathcal{K}^m_{\lambda,\mu,\alpha} f(z) \\ \mathcal{K}^{m,2}_{\lambda,\mu,\alpha} f(z) &= \mathcal{K}^m_{\lambda,\mu,\alpha}(\mathcal{K}^{m,1}_{\lambda,\mu,\alpha} f(z)) \\ &\cdots\cdots \\ &\cdots\cdots \\ &\cdots\cdots \\ \mathcal{K}^{m,s}_{\lambda,\mu,\alpha} f(z) &= \mathcal{K}^m_{\lambda,\mu,\alpha}(\mathcal{K}^{m,s-1}_{\lambda,\mu,\alpha} f(z)). \end{aligned} \tag{1.8}$$

If $f \in \mathcal{A}$ is given by (1.1), then making use of (1.7) and (1.8) we conclude that

$$\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z) = z + \sum_{n=2}^{\infty} \left[1 + \frac{\lambda(n-1)}{\mu+1}\right]^m [1+(n-1)\alpha]^s a_n z^n, \tag{1.9}$$

$$(m \in \mathbb{Z},\ s \in \mathbb{N} \cup \{0\},\ \alpha \geq 0,\ \mu > -1,\ \lambda > 0).$$

The operator $\mathcal{K}_{\lambda,\mu,\alpha}^{m,s}$ generalizes several previously studied familiar operators. The following are some of the interesting particular cases:

- For $\mu = 0, s = 0$, $\mathcal{K}_{\lambda,0,\alpha}^{m,0} = D_\lambda^m$ and $m = 0$, $\mathcal{K}_{\lambda,\mu,\alpha}^{0,s} = D_\alpha^s$ has been studied by Al-Oboudi [1] (also see [4, 10]);
- for $\lambda = 1, \mu = 0, s = 0$, $\mathcal{K}_{1,0,\alpha}^{m,0} = D^m$ and $m = 0, \alpha = 1$, $\mathcal{K}_{\lambda,\mu,1}^{0,s} = D^s$ has been studied by Sălăgean [12];
- for $\lambda = 1, \mu = a$ $(a \geq 0)$ and $m = -k$ $(k \in \mathbb{N}_0)$, the operator $\mathcal{K}_{1,a,\alpha}^{-k,s}$ reduces to L_{a+1}^k has been studied by Komatu [6].

It can be easily verified from (1.9) that

$$z\left(\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)\right)' = \frac{\mu+1}{\lambda}\left(\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)\right) + \left(1 - \frac{\mu+1}{\lambda}\right)\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z). \tag{1.10}$$

By making use of the operator $\mathcal{K}_{\lambda,\mu,\alpha}^{m,s}$, we define a new subclass of $\mathcal{A}$ as follows:

Definition 1.1 *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{\lambda,\mu}^*(m,s,\alpha;\eta)$ if it satisfies the following condition:*

$$\Re\left\{\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)}\right\} > \eta \quad (0 \leq \eta < 1; z \in \mathbb{U}). \tag{1.11}$$

In particular, for $m = 0, s = 0, \mu = 0, \lambda = 1$, the class $\mathcal{S}_{\lambda,\mu}^*(m,s,\alpha;\eta)$ reduces to $\mathcal{S}^*(\eta)$, the class of starlike functions of order η while for $m = 1, s = 0, \mu = 0, \lambda = 1$, the class $\mathcal{S}_{\lambda,\mu}^*(m,s,\alpha;\eta)$ reduces to $\mathcal{CV}(\eta)$, the class of convex function of order η.

In the present paper, the author determines sufficient conditions for a function $f \in \mathcal{A}$ to be a member of the class $\mathcal{S}_{\lambda,\mu}^*(m,s,\alpha;\eta)$. Some corollaries are also deduced from the main result. The concluding section describe the possibility of further extension of the present result to p-valent function.

In order to derive the main results we need the following lemma.

Lemma 1.2 *([5], also see [8]) Let the (non constant) function $w(z)$ be analytic in $\mathbb{U}$ with*

$$w(0) = 0 \quad \textit{and} \quad w(z) \neq 0 \quad (z \in \mathbb{U}). \tag{1.12}$$

If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then

$$z_0 w'(z_0) = \xi w(z_0) \tag{1.13}$$

where $\xi \geq 1$ is some real number.

2 Main Results

Theorem 2.1 *If $f \in \mathcal{A}$ satisfies the following condition:*

$$\left|\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)} - 1\right|^{\gamma} \left|\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+2,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)} - 1\right|^{\delta} < M(\lambda,\mu,\delta,\gamma;\eta) \quad (z \in \mathbb{U}) \tag{2.1}$$

for some real numbers $\delta,\ \gamma,\ \eta$ such that $0 \leq \eta < 1,\ \delta+\gamma \geq 0,\ m \in \mathbb{Z},\ s \in \mathbb{N}_0,\ \lambda > 0,\ \mu > -1,\ \alpha \geq 0$ then $f \in \mathcal{S}^*_{\lambda,\mu}(m,s,\alpha;\eta)$, where

$$M(\lambda,\mu,\delta,\gamma;\eta) = \begin{cases} (1-\eta)^{\gamma}\left(1-\eta+\frac{\lambda}{2(\mu+1)}\right)^{\delta} & (0 \leq \eta \leq \frac{1}{2}) \\ (1-\eta)^{\delta+\gamma}\left(1+\frac{\lambda}{\mu+1}\right)^{\delta} & (\frac{1}{2} \leq \eta < 1). \end{cases} \tag{2.2}$$

Proof: Case(i) Let $0 \leq \eta \leq \frac{1}{2}$. Define a function $w(z)$ as

$$\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)} = \frac{1+(1-2\eta)w(z)}{1-w(z)} \quad (w(z) \neq 1; z \in \mathbb{U}). \tag{2.3}$$

Clearly, $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$ in $\mathbb{U}$. Differentiating logarithmically on both sides of (2.3) with respect to z and multiplying the resulting equation by z, we obtain

$$\frac{z\left(\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)\right)'}{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)} - \frac{z\left(\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)\right)'}{\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)} = \frac{(1-2\eta)zw'(z)}{1+(1-2\eta)w(z)} + \frac{zw'(z)}{1-w(z)}. \tag{2.4}$$

Using the identity (1.10) in (2.4), we have

$$\frac{\mu+1}{\lambda}\left[\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+2,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)} - \frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)}\right] = \frac{2(1-\eta)zw'(z)}{[1-w(z)][1+(1-2\eta)w(z)]},$$

which implies

$$\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+2,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)} - 1 = \frac{2(1-\eta)w(z)}{1-w(z)} + \frac{2\lambda(1-\eta)zw'(z)}{(1+\mu)(1-w(z))(1+(1-2\eta)w(z))}. \tag{2.5}$$

Also from (2.3), it is deduce that

$$\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)} - 1 = \frac{2(1-\eta)w(z)}{1-w(z)}. \tag{2.6}$$

From (2.5) and (2.6), it follows that

$$\begin{aligned} &\left|\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m,s} f(z)} - 1\right|^{\gamma} \left|\frac{\mathcal{K}_{\lambda,\mu,\alpha}^{m+2,s} f(z)}{\mathcal{K}_{\lambda,\mu,\alpha}^{m+1,s} f(z)} - 1\right|^{\delta} \\ &= \left|\frac{2(1-\eta)w(z)}{1-w(z)}\right|^{\gamma} \left|\frac{2(1-\eta)w(z)}{1-w(z)} + \frac{2\lambda(1-\eta)zw'(z)}{(1+\mu)[1-w(z)][1+(1-2\eta)w(z)]}\right|^{\delta} \\ &= \left|\frac{2(1-\eta)w(z)}{1-w(z)}\right|^{\delta+\gamma} \left|1 + \frac{\lambda zw'(z)}{(1+\mu)[1+(1-2\eta)w(z)]w(z)}\right|^{\delta}. \end{aligned} \tag{2.7}$$

Now suppose that there exist a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1.$$

Then from Lemma 1.2 we have $w(z_0) = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = \xi w(z_0)(\ \xi \geq 1)$. Therefore,

$$\begin{aligned}
&\left|\frac{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha} f(z_0)}{\mathcal{K}^{m,s}_{\lambda,\mu,\alpha} f(z_0)} - 1\right|^{\gamma} \left|\frac{\mathcal{K}^{m+2,s}_{\lambda,\mu,\alpha} f(z_0)}{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha} f(z_0)} - 1\right|^{\delta} \\
&= \left|\frac{2(1-\eta)w(z_0)}{1-w(z_0)}\right|^{\delta+\gamma} \left|1 + \lambda \frac{z_0 w'(z_0)}{(1+\mu)[1+(1-2\eta)w(z_0)]w(z_0)}\right|^{\delta} \\
&= \frac{2^{\delta+\gamma}(1-\eta)^{\delta+\gamma}}{|1-e^{i\theta}|^{\delta+\gamma}} \left|1 + \frac{\lambda\xi}{(1+\mu)[1+(1-2\eta)e^{i\theta}]}\right|^{\delta} \\
&\geq (1-\eta)^{\delta+\gamma} \left[1 + \frac{\lambda\xi}{2(1+\mu)(1-\eta)}\right]^{\delta} \\
&\geq (1-\eta)^{\delta+\gamma} \left(1 + \frac{\lambda}{2(1+\mu)(1-\eta)}\right)^{\delta} \quad (\xi \geq 1) \\
&= (1-\eta)^{\gamma} \left(1 - \eta + \frac{\lambda}{2(1+\mu)}\right)^{\delta} \\
&= M(\lambda, \mu, \delta, \gamma; \eta)
\end{aligned} \tag{2.8}$$

which contradicts our assumption (2.1) for $0 \leq \eta \leq \frac{1}{2}$. Therefore, $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$. Hence $f \in \mathcal{S}^*_{\lambda,\mu}(m, s, \alpha; \eta)$.

Case(ii) Let $\frac{1}{2} \leq \eta < 1$. Define the function $w(z)$ by

$$\frac{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha} f(z)}{\mathcal{K}^{m,s}_{\lambda,\mu,\alpha} f(z)} = \frac{\eta}{\eta - (1-\eta)w(z)} \quad (z \in \mathbb{U}) \tag{2.9}$$

where $w(z) \neq \frac{\eta}{1-\eta}$ in $\mathbb{U}$. Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Taking logarithmic Differentiation on both sides of (2.9) with respect to z and using the identity (1.10) in the resulting equation, we obtain

$$\frac{\mathcal{K}^{m+2,s}_{\lambda,\mu,\alpha} f(z)}{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha} f(z)} = \frac{\lambda(1-\eta)zw'(z)}{(1+\mu)[\eta-(1-\eta)w(z)]} + \frac{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha} f(z)}{\mathcal{K}^{m,s}_{\lambda,\mu,\alpha} f(z)}. \tag{2.10}$$

Making use of (2.9) in (2.10) and after simplification gives

$$\frac{\mathcal{K}^{m+2,s}_{\lambda,\mu,\alpha} f(z)}{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha} f(z)} - 1 = \frac{(1-\eta)w(z)}{\eta-(1-\eta)w(z)} \left[1 + \frac{\lambda z w'(z)}{(1+\mu)w(z)}\right]. \tag{2.11}$$

Also, it follows from (2.9) that

$$\frac{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha} f(z)}{\mathcal{K}^{m,s}_{\lambda,\mu,\alpha} f(z)} - 1 = \frac{(1-\eta)w(z)}{\eta-(1-\eta)w(z)}. \tag{2.12}$$

Therefore from (2.11) and (2.12), we obtain

$$\left|\frac{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha}f(z)}{\mathcal{K}^{m,s}_{\lambda,\mu,\alpha}f(z)}-1\right|^{\gamma}\left|\frac{\mathcal{K}^{m+2,s}_{\lambda,\mu,\alpha}f(z)}{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha}f(z)}-1\right|^{\delta}=\left|\frac{(1-\eta)w(z)}{\eta-(1-\eta)w(z)}\right|^{\delta+\gamma}\left|1+\frac{\lambda}{1+\mu}\frac{zw'(z)}{w(z)}\right|^{\delta}. \tag{2.13}$$

Suppose that there exist a point $z_0 \in \mathbb{U}$ such that $\max_{|z|<|z_0|}|w(z)| = |w(z_0)| = 1$. Then by application of Lemma 1.2 we obtain $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = \xi w(z_0)$ $(\xi \geq 1)$.
Now

$$\begin{aligned}\left|\frac{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha}f(z)}{\mathcal{K}^{m,s}_{\lambda,\mu,\alpha}f(z)}-1\right|^{\gamma}\left|\frac{\mathcal{K}^{m+2,s}_{\lambda,\mu,\alpha}f(z)}{\mathcal{K}^{m+1,s}_{\lambda,\mu,\alpha}f(z)}-1\right|^{\delta}&=\left|\frac{(1-\eta)w(z_0)}{\eta-(1-\eta)w(z_0)}\right|^{\delta+\gamma}\left|1+\frac{\lambda}{(1+\mu)}\frac{z_0w'(z_0)}{w(z_0)}\right|^{\delta}\\&\geq(1-\eta)^{\delta+\gamma}\left(1+\frac{\lambda}{1+\mu}\right)^{\delta}\\&=M(\lambda,\mu,\delta,\gamma;\eta)\end{aligned} \tag{2.14}$$

which contradicts the assumption of (2.1) for $\frac{1}{2} \leq \eta < 1$. Therefore, we must have $|w(z)| < 1$ for all $z \in \mathbb{U}$ which implies $f \in \mathcal{S}^*_{\lambda,\mu}(m, s, \alpha; \eta)$. Thus, the proof of Theorem 2.1 is completed.

Letting $m = s = \mu = 0$ and $\lambda = 1$ in Theorem 2.1 give

Corollary 2.2 *If $f \in \mathcal{A}$ satisfies the following condition:*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\delta} < M(\delta,\gamma;\eta) \quad (z \in \mathbb{U})$$

for some real numbers η, δ, γ such that $0 \leq \eta < 1$, $\delta + \gamma \geq 0$, then $f \in \mathcal{S}^*(\eta)$ where

$$M(\delta,\gamma;\eta)=\begin{cases}(1-\eta)^{\gamma}(\frac{3}{2}-\eta)^{\delta} & (0\leq\eta\leq\frac{1}{2})\\ 2^{\delta}(1-\eta)^{\delta+\gamma} & (\frac{1}{2}\leq\eta<1).\end{cases}$$

Putting $\eta = \frac{1}{2}$ in the Corollary 2.2 we get:

Corollary 2.3 *If $f \in \mathcal{A}$ satisfies the condition:*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\delta} < \frac{1}{2^{\gamma}} \quad (z \in \mathbb{U})$$

for some real numbers δ, γ such that $\delta + \gamma \geq 0$, then $f \in \mathcal{S}^*(\frac{1}{2})$.

Taking $m = 1$, $s = \mu = 0$ and $\lambda = 1$ in Theorem 2.1 give

Corollary 2.4 *If $f \in \mathcal{A}$ satisfies the condition:*

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\gamma}\left|\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right|^{\delta} < M(\delta,\gamma;\eta) \quad (z \in \mathbb{U})$$

for some real numbers η, δ, γ *such that* $0 \le \eta < 1$, $\delta + \gamma \ge 0$, *then* $f \in \mathcal{CV}(\eta)$ *where*

$$M(\delta, \gamma; \eta) = \begin{cases} (1-\eta)^{\gamma}(\frac{3}{2}-\eta)^{\delta} & (0 \le \eta \le \frac{1}{2}) \\ 2^{\delta}(1-\eta)^{\delta+\gamma} & (\frac{1}{2} \le \eta < 1). \end{cases}$$

Further, putting $\eta = \frac{1}{2}$ in Corollary 2.4 we obtain the following result.

Corollary 2.5 *If* $f \in \mathcal{A}$ *satisfies the condition:*

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\gamma} \left|\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right|^{\delta} < \frac{1}{2^{\gamma}} \quad (z \in \mathbb{U})$$

for some real numbers δ, γ *such that* $\delta + \gamma \ge 0$, *then* $f \in \mathcal{CV}(\frac{1}{2})$.

3 Concluding Section

In this section, we extend the present work for p-valently analytic functions instead of univalent analytic functions defined as (1.1). Let $\mathcal{A}_p$ be denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (n \in \mathbb{N}, z \in \mathbb{U}) \tag{3.1}$$

that are analytic and p-valent functions in $\mathbb{U}$.

For $\mu > -1, \lambda \ge 0, p \in \mathbb{N}$, define an integral operator $\mathcal{I}_{p,\lambda,\mu} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ by

$$\mathcal{I}_{p,\lambda,\mu} f(z) = \begin{cases} \frac{\mu+p}{\lambda} z^{p-\frac{\mu+p}{\lambda}} \int_0^z t^{\frac{\mu+p}{\lambda}-p-1} f(t)dt & (\lambda \ne 0) \\ f(z) & (\lambda = 0), \end{cases} \tag{3.2}$$

and a differential operator $\mathcal{D}_{p,\lambda,\mu} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ by

$$\mathcal{D}_{p,\lambda,\mu} f(z) = \begin{cases} \frac{\lambda}{\mu+1} z^{1+p-\frac{\mu+p}{\lambda}} \frac{d}{dz}\left(z^{\frac{\mu+p}{\lambda}-p} f(z)\right) & (\lambda \ne 0) \\ f(z) & (\lambda = 0). \end{cases} \tag{3.3}$$

For $p \in \mathbb{N}, m \in \mathbb{Z}, \mu > -p, \lambda > 0$, define a linear operator $\mathcal{T}^m_{p,\lambda,\mu} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ as

$$\mathcal{T}^m_{p,\lambda,\mu} f(z) = \begin{cases} \mathcal{I}_{p,\lambda,\mu} \mathcal{T}^{m+1}_{p,\lambda,\mu} f(z) & (m \in \mathbb{Z}^-) \\ \mathcal{D}_{p,\lambda,\mu} \mathcal{T}^{m-1}_{p,\lambda,\mu} f(z) & (m \in \mathbb{Z}^+) \\ f(z) & (m = 0). \end{cases} \tag{3.4}$$

Thus, for a function $f \in \mathcal{A}_p$ given by (3.1), it follows from (3.4) that

$$\mathcal{T}^m_{p,\lambda,\mu} f(z) = z^p + \sum_{n=p+1}^{\infty} \left[1 + \frac{\lambda(n-p)}{\mu+p}\right]^m a_n z^n. \tag{3.5}$$

Corresponding to the operator $\mathcal{T}^m_{p,\lambda,\mu}$, we define the linear multiplier operator $\mathcal{K}^{m,s}_{p,\lambda,\mu,\alpha}$: $\mathcal{A}_p \longrightarrow \mathcal{A}_p$ by

$$\begin{aligned}
\mathcal{K}^{m,0}_{p,\lambda,\mu,\alpha}f(z) =&\mathcal{T}^m_{p,\lambda,\mu}f(z),\\
\mathcal{K}^{m,1}_{p,\lambda,\mu,\alpha}f(z) =&(1-\alpha)\mathcal{T}^m_{p,\lambda,\mu}f(z) + \frac{\alpha z}{p}\left(\mathcal{T}^m_{p,\lambda,\mu}f(z)\right)' = \mathcal{K}_{p,\lambda,\mu,\alpha}f(z),\\
\mathcal{K}^{m,2}_{p,\lambda,\mu,\alpha}f(z) =&\mathcal{K}^m_{p,\lambda,\mu,\alpha}\left(\mathcal{K}^{m,1}_{p,\lambda,\mu,\alpha}f(z)\right)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\mathcal{K}^{m,s}_{p,\lambda,\mu,\alpha}f(z) =&\mathcal{K}^m_{p,\lambda,\mu,\alpha}\left(\mathcal{K}^{m,s-1}_{p,\lambda,\mu,\alpha}f(z)\right).
\end{aligned} \tag{3.6}$$

Thus, for a function $f \in \mathcal{A}_p$ given by (3.1), it is deduce from (3.6) that

$$f(z) = z^p + \sum_{n=p+1}^{\infty}\left[1+\frac{\lambda(n-p)}{\mu+p}\right]^m\left[1+\frac{\alpha}{p}(n-p)\right]^s a_n z^n \quad (z \in \mathbb{U}). \tag{3.7}$$

Using the operator $\mathcal{K}^{m,s}_{p,\lambda,\mu,\alpha}$, one can define the class $\mathcal{S}^*_{p,\lambda,\mu}(m,s,\alpha;\eta)$ similar to that of (1.11).

A good amount of literature available for finding the various properties of univalent (or multivalent) analytic functions containing several types of linear multiplier operators associated with the simple form of operations of differentiation and integration (for details, see [2, 7, 9, 13]). It left as an open problem for researcher to find out the sufficient conditions for a function $f \in \mathcal{A}_p$ is in class $\mathcal{S}^*_{p,\lambda,\mu}(m,s,\alpha;\eta)$ and compare the result of Theorem 2.1 when $p = 1$.

References

[1] F.M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, Int. J. Math. Math. Sci., 27(2004), 1429-1436.

[2] S. Ding, Y. Ling and G. Bao, Some properties of a class of analytic functions, J. Math. Anal. Appl., 195(1995), 71-81.

[3] P.L. Duren, Univalent Functions, Graduate Texts in Mathematics, Springer-Verlag, New York, (1983), 259.

[4] S.S. Eker and H.O. Güney, A new subclass of analytic functions involving Al-Oboudi differential operator, J. Inequal. Appl., (2008), 10 pages.

[5] I.S. Jack, Functions starlike and convex of order α, J. London Math. Soc.(ser 2), 3(1971), 469-474.

[6] Y. Komatu, On analytical prolongation of a family of integral operators, Mathematica (Cluj), 32(1990), 141-145.

[7] A.Y. Lashin, Some convolution properties of analytic functions, Appl. Math. Lett., 18(2005), 135-138.

[8] S.S. Miller and P.T. Mocanu, Second-order differential inequalities in the complex plane, J. Math. Anal. Appl., 65(1978), 289-305.

[9] O. Ozkan, Some subordination results on multivalent functions defined by integral operator, J. Inequal. Appl., 71616(2007), 1-8.

[10] A. Rahman, S. Juma and S.R. Kulkarni, On univalent functions with negative coefficients by using generalized S*ălă*gean operator, Filomat, 21(2) (2007), 173-184.

[11] M.S. Robertson, On the theory of univalent functions, Ann. Math., 37(1936), 374-408.

[12] G.S. S*ălă*gean, Subclasses of univalent functions, In: Complex Analysis: Fifth Romanian Finish Seminar, Part-I (Bucharest, 1981), Lecture Notes in Math., Springer- Verlag, 1013(1983), 362-372.

[13] T.M. Seoudy, On certain subclass of p-valent functions defined by the the Jung-Kim-Srivastava integral operator, Electronic J. Math Anal. Appl., 1(2) (2013), 368-373.
