Information in **S**ciences and **C**omputing
Volume 2014, Number 3, Article ID ISC370814, 05 pages
Available online at http://www.infoscicomp.com/

Research Article

The Linear Complementarity Problem and a Method to Find all its Solutions

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Received 15 May 2014; Accepted 20 June 2014

Abstract: To solve a linear complementarity problem LCP, almost all the authors assume that this problem admits one and only one solution, and they give a method to calculate this solution. The aim of this paper is to solve LCP when this problem has several solutions.

Keywords: Linear complementarity problem; Linear system; All solutions; Principal submatrix.

1 Introduction

The linear complementarity problem is to find a vector x in IR^n satisfying $x \geqslant 0$, $Ax + b \geqslant 0$ and $x^T(Ax + b) = 0$, or showing that no such vector exists, where $A \in \mathrm{IR}^{n\times n}$ and $b \in \mathrm{IR}^n$ are given data.

The linear complementarity problem, denoted by $LCP(A, b)$, introduced by Cottle [3], is one of the most widely studied mathematical programming problems. Solving $LCP(A, b)$ for an arbitrary matrix A is NP-complete [2], while there are several classes of matrices A for which the associated $LCPs$ can be solved efficiently. For details of the theory of $LCPs$, see the books of Cottle et al. [4] and Murty [14].

To solve the linear complementarity problem $LCP(A, b)$, almost all the authors assume that this problem admits one and only one solution (they impose certain conditions on the matrix A, in particular, positive-definite matrix, P-matrix, etc.), and they give an iterative method to calculate this solution (see [1,5–13,15–18], and many other papers), but the question is, when LCP has several solutions, how to calculate them?

2 Preliminaries

The main result of this paper is to solve LCP where A is an arbitrary matrix. In order to be able to do this, we first need some notation. We denote by $[n]$ the set of integers from 1 to n, and we will need the following definitions.

Definition 2.1 *Given $A \in \mathrm{IR}^{n\times n}$, let X, Y be subsets of $[n]$. $A[X,Y]$ denotes the submatrix of A having rows indexed by elements of X and columns indexed by elements of Y.*

Definition 2.2 *Given $b \in \mathrm{IR}^n$, let X be a subset of $[n]$. $b[X]$ denotes the subvector of b with rows indexed by elements of X.*

We shall denote by $A[X]$ the principal submatrix $A[X,X]$, and we denote the complement of X in the set $[n]$ by $\bar{X}$.

3 The Main Result

The key in our approach to finding all solutions to LCP is the following theorem.

Theorem 3.1 *For any matrix $A \in \mathrm{IR}^{n\times n}$ and vector $b \in \mathrm{IR}^n$, $LCP(A,b)$ has a solution if and only if there is a subset $X \subseteq [n]$ satisfying $A[X]x + b[X] = 0$ has a nonnegative solution $x^*[X] \geqslant 0$, and $A[\bar{X},X]x^*[X] + b[\bar{X}] \geqslant 0$.*

Proof: Let x^* be a solution of the $LCP(A,b)$, that is: $x^* \geqslant 0$, $y^* = Ax^* + b \geqslant 0$ and $x^{*T}y^* = 0$. Hence $x_i^* = 0$ or $y_i^* = 0$ for all $i \in [n]$. Now, let $X = \{i \in [n] : x_i^* > 0\}$ and let $\overline{X} = \{i \in [n] : x_i^* = 0\}$ the complement of X in the set $[n]$. However, there are two possible cases. The first one: $X = \emptyset$, that is, $x^* = 0$ and $y^* = Ax^* + b = b$. This case is possible only if $b \geqslant 0$, then $x^* = 0$ is a solution of $LCP(A,b)$. The second one: $X \neq \emptyset$, this means that, for all $i \in X$ we have $0 = y_i^* = \sum_{j\in X} a_{ij}x_j^* + b_i$ or in matrix form, $A[X]x^*[X] + b[X] = 0$; now for all $i \in \bar{X}$, we have $y_i^* = \sum_{j\in X} a_{ij}x_j^* + b_i$ or in matrix form $y^*[\bar{X}] = A[\bar{X},X]x^*[X] + b[\bar{X}]$. Thus, in both cases, there is a subset X such that the system of linear equations $A[X]x + b[X] = 0$ has a nonnegative solution $x^*[X] \geqslant 0$ and $y^*[\bar{X}] = A[\bar{X},X]x^*[X] + b[\bar{X}] \geqslant 0$ with conventionally $x[\emptyset] = 0$.

Now we assume that there is a subset $X \subseteq [n]$ such that the system of linear equations $A[X]x + b[X] = 0$ has a nonnegative solution $x \geqslant 0$ and $y = A[\bar{X},X]x + b[\bar{X}] \geqslant 0$.

Let $x^* = (x_i^*)_{i\in[n]}$ and $y^* = (y_i^*)_{i\in[n]}$ where $x_i^* = \begin{cases} x_i & \text{if } i \in X \\ 0 & \text{if } i \in \bar{X} \end{cases}$ and $y_i^* = \begin{cases} 0 & \text{if } i \in X \\ y_i & \text{if } i \in \bar{X} \end{cases}$, it can be easily to verified that $x^* \geqslant 0$, $y^* \geqslant 0$ and $x^{*T}y^* = 0$ and therefore x^* is a solution of the $LCP(A,b)$. This concludes the proof.

Note that if there is a unique subset $X \subseteq [n]$ satisfying (a) the matrix $A[X]$ is invertible; (b) $(A[X])^{-1}b[X] \leqslant 0$; (c) and $A[\bar{X},X](A[X])^{-1}b[X] - b[\bar{X}] \leqslant 0$. Then $LCP(A,b)$ has a unique solution.

Now, we denote the set of all subsets of $[n]$ by $\mathrm{S}_n = \{X \text{ such that } X \subseteq [n]\}$. It is known that the cardinality of the S_n is 2^n. We assume that the set S_n is sorted in increasing order (e.g., if $n = 3$, then
$\mathrm{S}_3 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$). Finally, we denote the set of solutions of $LCP(A, b)$ by S.
The next step is to give the following algorithm to compute the set S.

Algorithm
For $k = 0$ to $2^n - 1$ do the following.
Let $X_k \in \mathrm{S}_n$. Recall that the cardinality of the empty set is 0.
Let x be a solution of the system of linear equations $A[X_k]x + b[X_k] = 0$.
If $x \geqslant 0$ and $y[\bar{X}_k] = A[\bar{X}_k, X_k]x + b[\bar{X}_k] \geqslant 0$, then $x^* \in \mathrm{S}$, where $x^*[X_k] = x$ and $x^*[\bar{X}_k] = 0$;
else $k \longleftarrow k + 1$.

4 Numerical Example

To illustrate the result we will take the following simple example which admits four solutions.

Let us consider the following linear complementarity problem LCP, find vector x satisfying $x \geqslant 0$, $Ax + b \geqslant 0$ and $x^T(Ax + b) = 0$,

where $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -3 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

The following table shows the different solutions of this problem. Column 1 and 2 show the step k and the subset X_k, respectively; column 3 (resp. 4) reports the solution x to the system of equations $A[X_k]x + b[X_k] = 0$ (resp. the vector $y[\bar{X}_k] = A[\bar{X}_k, X_k]x + b[\bar{X}_k]$); column 5 presents the solution x^* to LCP (when it exists); and column 6 is the vector $y^* = Ax^* + b$ (when it exists).

k	X_k	$x^*[X_k]$	$y^*[\bar{X}_k]$	x^*	y^*
0	$\emptyset$	$(0,0,0)^T$	$(1,-2,3)^T$	*****	*****
1	$\{1\}$	$\frac{1}{3}$	$\left(\frac{-5}{3}, \frac{11}{3}\right)^T$	*****	*****
2	$\{2\}$	$\frac{2}{3}$	$\left(\frac{5}{3}, \frac{11}{3}\right)^T$	$(0, \frac{2}{3}, 0)^T$	$\left(\frac{5}{3}, 0, \frac{11}{3}\right)^T$
3	$\{3\}$	1	$(3,-1)^T$	*****	*****
4	$\{1,2\}$	$(\frac{1}{2}, \frac{1}{2})^T$	$\frac{9}{2}$	$(\frac{1}{2}, \frac{1}{2}, 0)^T$	$\left(0, 0, \frac{9}{2}\right)^T$
5	$\{1,3\}$	$(\frac{9}{5}, \frac{11}{5})^T$	2	$(\frac{9}{5}, 0, \frac{11}{5})^T$	$(0,2,0)^T$
6	$\{2,3\}$	$(\frac{3}{10}, \frac{11}{10})^T$	$\frac{7}{2}$	$(0, \frac{3}{10}, \frac{11}{10})^T$	$\left(\frac{7}{2}, 0, 0\right)^T$
7	$\{1,2,3\}$	$(\frac{7}{5}, \frac{-2}{5}, \frac{9}{5})^T$	*****	*****	*****

For $k = 0, 1, 3, 7$, it is easy to show that in each case, whether the system $A[X_k]x + b[X_k] = 0$ does not have a nonnegative solution $x^*[X_k] \ngeqslant 0$, or $y^*[\bar{X}_k] = A[\bar{X}, X]x^*[X_k] + b[\bar{X}] \ngeqslant 0$. However, for $k = 2, 4, 5, 6$, the system of linear equations $A[X_k]x + b[X_k] = 0$ has a nonnegative solution $x^*[X_k] \geqslant 0$, and $y^*[\bar{X}_k] = A[\bar{X}_k, X_k]x^*[X_k] + b[\bar{X}_k] \geqslant 0$.
Thus, $\mathrm{S} = \left\{(0, \frac{2}{3}, 0)^T, (\frac{1}{2}, \frac{1}{2}, 0)^T, (\frac{9}{5}, 0, \frac{11}{5})^T, (0, \frac{3}{10}, \frac{11}{10})^T\right\}$.

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