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*Research Article*

# A Numerical Method for Solving Systems of Fredholm Integral Equations by Collocation Linear Legendre Multi-Wavelets

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**Abstract:** In this paper continuous Legendre multi-wavelets on the interval [0, 1) are utilized as a basis in collocation method to approximate the solutions of the Fredholm integral equations system. To begin with we describe the characteristic of Legendre multi-wavelets and will go on to indicate that through this method a system of Fredholm integral equations can be reduced to an algebraic equation. Finally, numerical results are given which support the theoretical results.

**Keywords:** Fredholm integral equation; System of integral equations; Legendre multi-wavelets; Collocation method; Multiresolution of analysis (MRA); Algebraic equations.

## 1. Introduction

Fredholm integral equations are classes of integral equations frequently encountered in applications. For such equations as well as a system of such equations, various techniques such as iterative, extrapolation, Galerkin, collocation, quadrature, projection, spline, orthogonal polynomial, and multiple grid methods have been presented to determine desired solutions (see e.g. [1-3] and the references quoted there). These methods include analytical, approximate, and numerical approaches. In principle, analytical solution is the most desired result in theory and it is almost unobtainable for most practical problems. Although numerical methods can cope with a majority of complicated problems related to a system of integral equations, the obtained results cannot be expressed in simple form. In comparison with numerical methods, one of the advantages of approximate methods lies in that it can give a solution in an analytic form with an allowable error. As a result, up-to-date approximate methods remain of much interest in spite of advanced numerical methods accompanied with the help of modern computers. Usual approximate methods include iterative methods, series expansion in terms of certain orthogonal functions, perturbation technique, and so on. Furthermore, the system of integral equations plays a basic role to many biological and engineering models. For instance, in several heat transfer problems in physics, the equations are usually replaced by system of integral equations [2]. Moreover, integro-differential equations, which are important in many practical problems, can be transformed to integral equations systems [4].

Let us consider the system of linear Fredholm integral equations of the form:

$$F(x) = G(x) + \int_{\Gamma} K(x,t)F(t)dt, \qquad x \in \Gamma = [0,1], \tag{1}$$

where,

$$F(x) = [f_1(x), f_2(x), ..., f_n(x)]^T,$$
$$G(x) = [g_1(x), g_2(x), ..., g_n(x)]^T,$$
$$K(x,t) = [k_{i,j}(x,t)]^T, \qquad i, j = 1,2,...,n.$$

In system (1) the known kernel *K(x, t)* is continuous, the function *G(x)* is given, and *F(x)* is the solution to be determined [4].

There have been considerable interests in solving integral equation (1). In addition to the well-known techniques, there are several new techniques for solving integral equation systems, such as Haar functions method[5], Adomian decomposition method[6], Block-Pulse functions[7], Rung-kutta method[8], Tau method[9], Newton-Taw method[10],Taylor collocation method[11], Sinc function basis [12], homotopy perturbation method[13], Biorthogonal systems method[14], triangular functions method[15], Fast multiscale Galerkin methods[16], reproducing kernel method[17]. As we know, it is important to select a suitable basis function in numerical methods for system of integral equations. One of the most attractive proposals made in the recent years was an idea connected to the application of wavelets as basis functions in the method of moments [18]. The wavelet technique allows the creation of very fast algorithms when compared to the algorithms ordinarily used and the main advantage of the wavelet technique is its ability to transform complex problems into a system of algebraic equations. Various wavelet basis are applied. In addition to the conventional Duabechies wavelets, Haar wavelets [19], linear B-splines [18], Walsh functions [20] have been used.

In this paper, we present the application of the linear Legendre multi-wavelets as basis functions in collocation's method for numerical solution of the system of Fredholm integral equations (1). The method is tested with the numerical examples.

## 2. Properties of Legendre Multi-Wavelets

Wavelet constitutes a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter *a* and the translation parameter *b* vary continuously, we have the following family of continuous wavelets as [21].

$$\varphi_{a,b}(t) = |a|^{-1/2} \phi(\frac{t-b}{a}), \qquad a, b \in \Re, a \neq 0.$$

If we restrict the parameters *a* and *b* to discrete values as $a = 2^{-k}, b = n2^{-k}$, then

$$\varphi_{k,n}(t) = 2^{-k/2}\phi(2^k t - n),$$

form an orthogonal basis[21].

The linear Legendre multi-wavelets are described in [22] and applied in [23-25]. For constructing the linear Legendre multi-wavelets, at first we describe the following scaling functions;

$$\varphi_0(t) = 1, \qquad \varphi_1(t) = \sqrt{3}(2t-1), \qquad 0 \le t \le 1.$$

Now let $\psi^0(t)$ and $\psi^1(t)$ be the corresponding mother wavelets, then by Multiresolution of analysis (MRA) and applying suitable conditions [22] on $\psi^0(t)$ and $\psi^1(t)$ the explicit formula for linear Legendre mother wavelets will obtain as;

$$\psi^0(t) = \begin{cases} -\sqrt{3}(4t-1), & 0 \le t \le \frac{1}{2}, \\ \sqrt{3}(4t-3), & \frac{1}{2} \le t \le 1, \end{cases} \tag{2}$$

$$\psi^1(t) = \begin{cases} 6t-1, & 0 \le t \le \frac{1}{2}, \\ 6t-5, & \frac{1}{2} \le t \le 1, \end{cases} \tag{3}$$

and the family $\{\psi^j_{k,n}\} = \{2^{k/2}\psi^j(2^k t - n)\}$ , $k$ is any nonnegative integer, $n = 0,1,...,2^k - 1$ and $j$ =0, 1, forms an orthonormal basis for $L^2(\Re)$ .

## 3. Function Approximation

A function $f(t)$ defined over [0, 1) may be expanded as;

$$f(t) = f_0\varphi_0(t) + f_1\varphi_1(t) + \sum_{k=0}^{\infty}\sum_{j=0}^{1}\sum_{n=0}^{\infty} f^j_{k,n}\psi^j_{k,n}(t), \tag{4}$$

Where,

$$f_0 =< f(t), \varphi_0(t) >, \qquad f_1 =< f(t), \varphi_1(t) >, \qquad f^j_{k,n} =< f(t), \psi^j_{k,n}(t) > . \tag{5}$$

In Eq. (5), $< .,. >$ denoting the inner product. If the infinite series of Eq. (4) is truncated, then it can be written as;

$$f(t) \approx f_0\varphi_0(t) + f_1\varphi_1(t) + \sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1} f^j_{k,n}\psi^j_{k,n}(t) = C^T\phi = \sum_{i=1}^{2M+2} c_i\phi_i(t), \tag{6}$$

where,

$$C = [f_0, f_1, f_{0,0}^0, f_{0,0}^1, \ldots, f_{M,0}^0, f_{M,1}^0, \ldots, f_{M,(2^M-1)}^0, \ldots, f_{M,0}^1, f_{M,1}^1, \ldots, f_{M,(2^M-1)}^1]^T,$$

$$\phi = [\varphi_0(t), \varphi_1(t), \psi_{0,0}^0(t), \psi_{0,0}^1(t), \ldots, \psi_{M,0}^0(t), \psi_{M,1}^0(t), \ldots, \psi_{M,(2^M-1)}^0(t), \ldots, \psi_{M,0}^1(t), \psi_{M,1}^1(t), \ldots, \psi_{M,(2^M-1)}^1(t)]^T,$$

and $M$ is a nonnegative integer.

## 4.4. Solving the System of Fredholm Integral Equations

In this section we apply collocation method to convert equation (1) to algebraic system of linear equations $AX = b$ and then solve this system by well known solver (see e.g. [26-30] and the references quoted there). We assume that Eq. (1) has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of system (1) could be found in [4].
we approximate $f_i(x)$'s, such that;

$$f_i(x) \approx \sum_{k=1}^{2M+2} c_{i,k}\phi_k(x), \qquad i = 1, \ldots, n. \tag{7}$$

where $\phi_k(x)$ defined in(6) and $c_{i,k}$'s are unknown coefficients which are determined by solving algebraic system of linear equations $AX = b$.

By substituting relation (7) in (1) we have;

$$\begin{cases} \sum_{k=1}^{2M+2} c_{1,k}\phi_k(x) = g_1(x) + \sum_{i=1}^{n} \int_\Gamma k_{1,i}(x,t) \sum_{k=1}^{2M+2} c_{i,k}\phi_k(t)dt, \\ \sum_{k=1}^{2M+2} c_{2,k}\phi_k(x) = g_2(x) + \sum_{i=1}^{n} \int_\Gamma k_{2,i}(x,t) \sum_{k=1}^{2M+2} c_{i,k}\phi_k(t)dt, \\ \vdots \\ \sum_{k=1}^{2M+2} c_{n,k}\phi_k(x) = g_n(x) + \sum_{i=1}^{n} \int_\Gamma k_{n,i}(x,t) \sum_{k=1}^{2M+2} c_{i,k}\phi_k(t)dt. \end{cases}$$

Now, we choose some collocation points such as;

$$x_i = \frac{i}{2M+2}, \qquad i = 1, \ldots, 2M+2,$$

Which are equidistant, also define system of residual equations by:

$$\begin{cases} E_1(x) = \sum_{k=1}^{2M+2} c_{1,k}\phi_k(x) - g_1(x) - \sum_{i=1}^{n} \int_\Gamma k_{1,i}(x,t) \sum_{k=1}^{2M+2} c_{i,k}\phi_k(t)dt, \\ E_2(x) = \sum_{k=1}^{2M+2} c_{2,k}\phi_k(x) - g_2(x) - \sum_{i=1}^{n} \int_\Gamma k_{2,i}(x,t) \sum_{k=1}^{2M+2} c_{i,k}\phi_k(t)dt, \\ \vdots \\ E_n(x) = \sum_{k=1}^{2M+2} c_{n,k}\phi_k(x) - g_n(x) - \sum_{i=1}^{n} \int_\Gamma k_{n,i}(x,t) \sum_{k=1}^{2M+2} c_{i,k}\phi_k(t)dt. \end{cases}$$

Then, by imposing the conditions;

$$E_i(x_j) = 0, \quad i = 1, \ldots, n \quad and \quad j = 1, \ldots, 2M + 2;$$

we can conclude algebraic system of linear equations $AX = b$ [1-2].

For example, for $n = 2$ we have;

$$\begin{cases} f_1(x) = g_1(x) + \int_\Gamma k_{1,1}(x,t) f_1(t)dt + \int_\Gamma k_{1,2}(x,t) f_2(t)dt, \\ f_2(x) = g_2(x) + \int_\Gamma k_{2,1}(x,t) f_1(t)dt + \int_\Gamma k_{2,2}(x,t) f_2(t)dt. \end{cases}$$

After discretization, algebraic system of linear equations $AX = b$ is concluded as follow;

$$A = (a_{i,j}), \quad i, j = 1, 2, \ldots, 4M + 4,$$

and,

$$b^T = [g_1(x_1), g_1(x_2), ..., g_1(x_{2M+2}), g_2(x_1), g_2(x_2), ..., g_2(x_{2M+2})],$$

$$X^T = [c_{1,1}, c_{1,2}, ..., c_{1,2M+2}, c_{2,1}, c_{2,2}, ..., c_{2,2M+2}],$$

$$a_{i,j} = \begin{cases} \phi_j(x_i) - \int_\Gamma k_{1,1}(x_i,t)\phi_j(t)dt, & for (i = 1,...,2M+2, \quad j = 1,...,2M+2), \\ -\int_\Gamma k_{1,2}(x_i,t)\phi_{j-2M-2}(t)dt, & for (i = 1,...,2M+2, \quad j = 2M+3,...,4M+4), \\ -\int_\Gamma k_{2,1}(x_{i-2M-2},t)\phi_j(t)dt, & for (i = 2M+3,...,4M+4, \quad j = 1,...,2M+2), \\ \phi_{j-2M-2}(x_{i-2M-2}) - \int_\Gamma k_{2,2}(x_{i-2M-2},t)\phi_{j-2M-2}(t)dt, & for (i = 2M+3,...,4M+4, \quad j = 2M+3,...,4M+4). \end{cases}$$

## 5. Numerical Experiments

In this section, we give some numerical experiments to illustrate the results obtained in previous sections. All the numerical experiments presented in this section were computed in double precision using a MATLAB 7 on a PC with a 1.86GHz 32-bit processor and 1GB memory.

**Example 5.1** Consider the following linear system of Fredholm integral equations:

$$\begin{cases} u(x) = \frac{x}{18} + \frac{17}{36} + \int_0^1 (\frac{x+t}{3})(u(t) + v(t))dt, \\ v(x) = x^2 - \frac{19x}{12} + 1 + \int_0^1 (\frac{x+t}{3})(u(t) + v(t))dt. \end{cases}$$

With the exact solutions $u(x) = x + 1$ and $v(x) = x + 1$.

The following table shows the numerical results of above example for M=1. In the Table 1, we reported the error of our method for different values of *x*.

**Table1:** Shows the results of example 5.1.

| $x$ | **Error for** $u(x)$ | **Error for** $v(x)$ |
|---|---|---|
| 0.0 | 0 | 0 |
| 0.1 | 0 | 0 |
| 0.2 | 0 | 0 |
| 0.3 | 0 | 0 |
| 0.4 | 0 | 0 |
| 0.5 | 2.220446049250313e-016 | 2.220446049250313e-016 |
| 0.6 | 4.440892098500626e-016 | 0 |
| 0.7 | 4.440892098500626e-016 | 0 |
| 0.8 | 6.661338147750939e-016 | 2.220446049250313e-016 |
| 0.9 | 4.440892098500626e-016 | 0 |
| 1.0 | 0 | 0 |

Furthermore, Plots of the exact and approximate solutions are presented in Fig. 1.

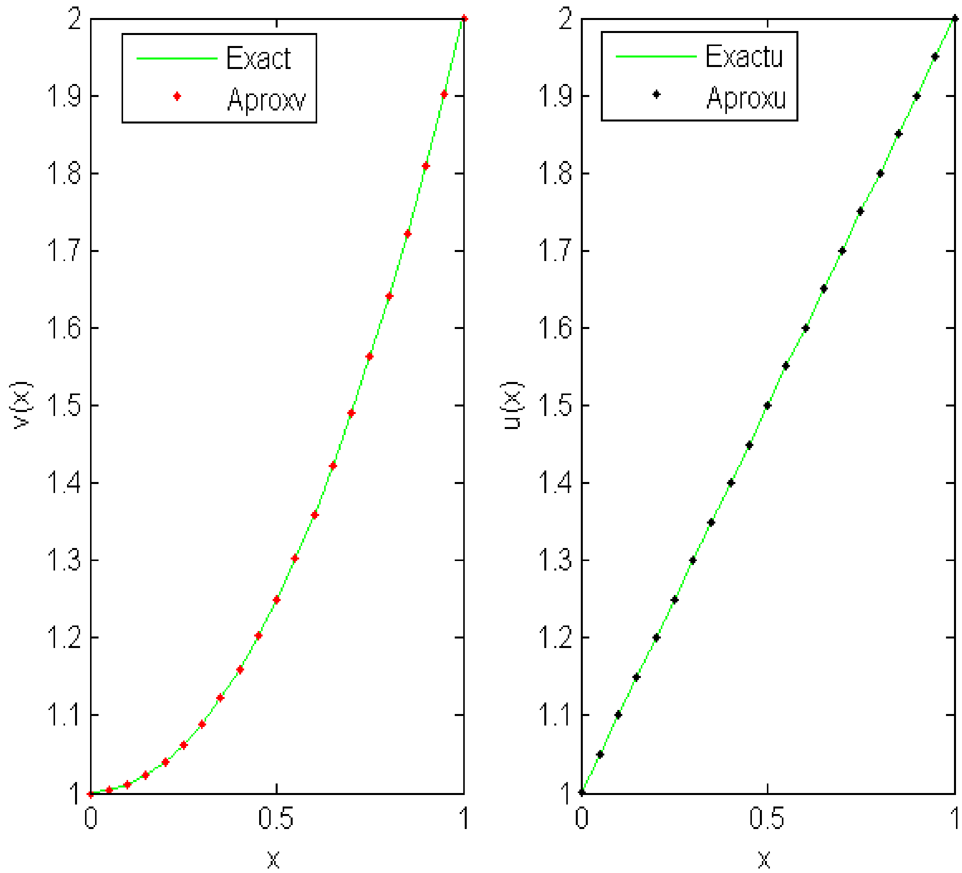


**Figure 1:** Comparison plot of exact and approximation solution of Example 5.1, for M=1

**Example 5.2** Consider the following system of Fredholm integral equations:

$$\begin{cases} u(x) = \dfrac{11x}{6} + \dfrac{11}{15} - \int_0^1 (x+t)u(t)dt - \int_0^1 (x+2t^2)v(t)dt, \\ v(x) = \dfrac{5}{4}x^2 + \dfrac{x}{4} - \int_0^1 (xt^2)u(t)dt - \int_0^1 (x^2 t)v(t)dt. \end{cases}$$

With the exact solutions $u(x) = x$ and $v(x) = x^2$.

For M=1,Table 2 and Fig. 2 are the numerical results for Example 5.2.

**Table 2:** Shows the results of example 5.2

| ***x*** | **Error for** $u(x)$ | **Error for** $v(x)$ |
|---|---|---|
| 0.0 | 6.757834038984984e-004 | 4.163336342344337e-017 |
| 0.1 | 7.958353679956648e-004 | 2.398293167558720e-003 |
| 0.2 | 2.401039281943351e-004 | 1.520341366488257e-003 |
| 0.3 | 1.035939296189914e-004 | 2.254068871461168e-003 |
| 0.4 | 4.802078563885592e-004 | 9.945748380517649e-003 |
| 0.5 | 6.002598204856935e-004 | 3.559888676329392e-003 |
| 0.6 | 7.203117845828277e-004 | 1.577299585309527e-004 |
| 0.7 | 1.645803447814842e-004 | 5.159817150472218e-004 |
| 0.8 | 9.604157127771407e-004 | 6.195281903595595e-003 |
| 0.9 | 4.046842729757305e-004 | 2.030307858454938e-003 |
| 1.0 | 5.007552358341283e-004 | 5.612794881025274e-003 |

**Table 3:** Shows the results of example 5.3

| **x** | **Error for** $u(x)$ | **Error for** $v(x)$ |
|---|---|---|
| 0.0 | 1.049890671469811e-004 | 1.110223024625157e-016 |
| 0.1 | 6.072925577794611e-003 | 3.201774443312155e-006 |
| 0.2 | 2.121606692097009e-003 | 1.364690746305519e-006 |
| 0.3 | 1.857946563313915e-003 | 5.511251090917213e-006 |
| 0.4 | 5.870270197581298e-003 | 1.742605106843653e-005 |
| 0.5 | 1.749721955872863e-004 | 2.808113651081579e-005 |
| 0.6 | 6.283257525539547e-003 | 4.125565102344186e-005 |
| 0.7 | 2.256086389597694e-003 | 6.576759635161800e-005 |
| 0.8 | 1.817635352255311e-003 | 9.531839982004353e-005 |
| 0.9 | 5.945386314777847e-003 | 1.299080614285963e-004 |
| 1.0 | 1.894959081804158e-003 | 4.992782885476170e-004 |

**Example 5.3** Consider the following system of Fredholm integral equations:

$$\begin{cases} u(x) = \dfrac{2e^x}{3} - \dfrac{1}{4} + \int_0^1 (\dfrac{1}{3}e^x t)u(t)dt + \int_0^1 t^2 v(t)dt, \\ v(x) = \dfrac{3}{2}x - x^2 + \int_0^1 (x^2 e^{-t})u(t)dt - \int_0^1 xv(t)dt. \end{cases}$$

With the exact solutions $u(x) = e^x$ and $v(x) = x$.

For M=2, Table 3 and Fig. 3 are the numerical results for Example 5.3

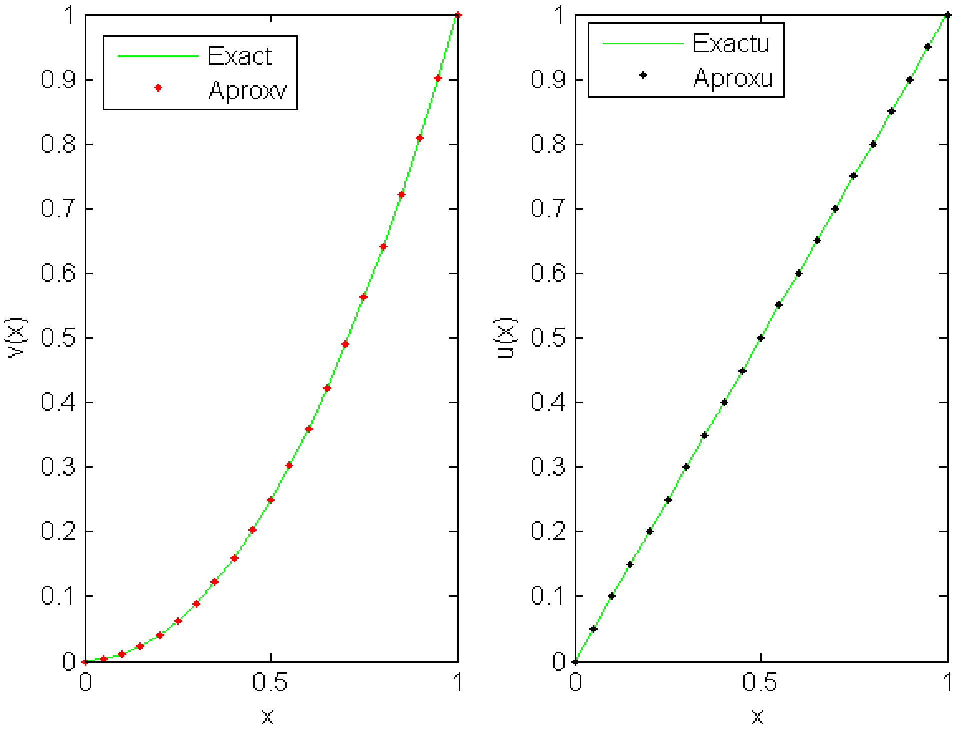


**Figure 2:** Comparison plot of exact and approximation solution of Example 5.2, for M=1

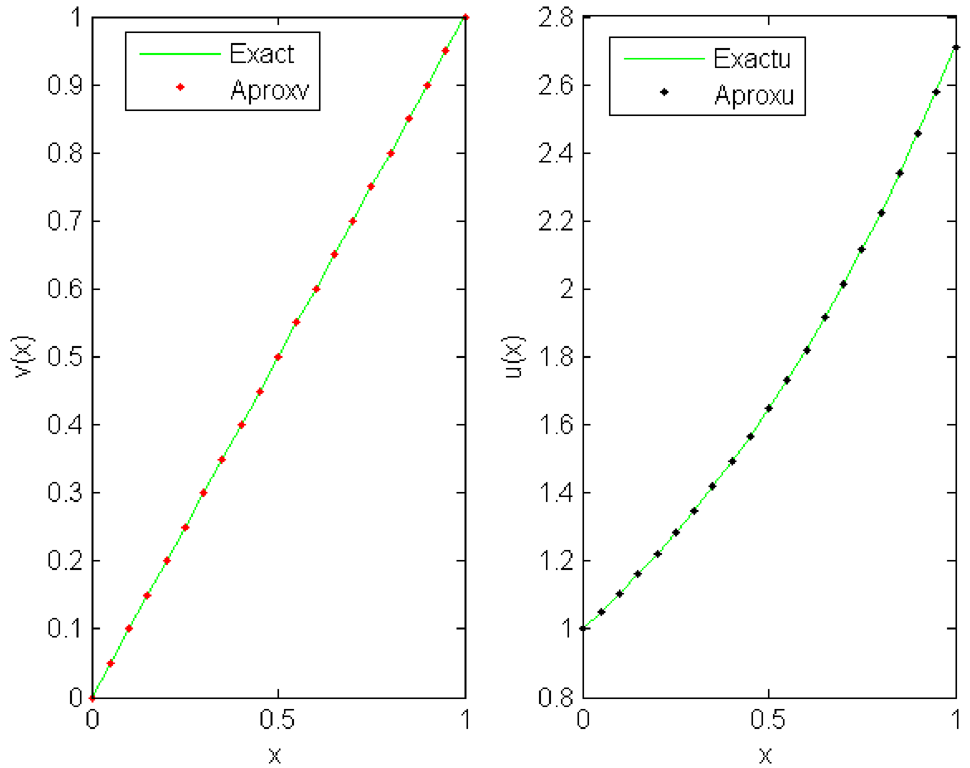


**Figure 3:** Comparison plot of exact and approximation solution of Example 5.3, for M=2

## 6. Conclusions

In this paper, the systems of Fredholm integral equations are investigated and a practical projection method known as collocation method based on Legendre multi-wavelets is proposed. The proposed method is easy to understand and this approximation reduces the system of integral equations to an explicit system of algebraic equations. Finally, Illustrative examples are included to demonstrate the validity and applicability of the technique.

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