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Research Article

Noether's Theorem in Riemannian Spaces

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Abstract: In arbitrary Riemannian 4-spaces, we construct continuity equations which could be interpreted as conservation laws for the energy and momentum of the gravitational field. We put special attention in general relativity.

Keywords: Noether's theorem; Lagrangians in curved spaces; Energy and momentum in Riemannian spaces; Rund-Lovelock's identities.

1. Introduction

Since general relativity is a physical theory the question of energy and momentum arises in natural way, and it has been the subject of considerable investigation dating back almost to the inception of the theory itself. Attempts at identifying an energy-momentum density for gravity, however, led only to various energy-momentum complexes which are not covariant objects (pseudotensors) because they inherently depend on the reference frame, and thus by their very nature cannot provide a truly physical local gravitational energy-momentum density. Indeed any such quantity is precluded by the equivalence principle itself, since a gravitational field should not be detectable at a point. Gravitation must, however, make a contribution to the energy of a system since, for example, the mass of a star is less than the sum of the rest masses of its individual particles. The principal aim of this work is to use the Noether's theorem to obtain, in a unified form, several important results on gravitational energy-momentum quantities scattered in the literature which were deduced through different methods.

For Lagrangians-based theories we exploit from very beginning the transformation properties of fields. In this paper, we consider gravitational Lagrangians:

$$L = L\left(g_{ab}; g_{ab,c}; g_{ab,cd}\right) \quad , \tag{1}$$

where g_{ij} is the metric tensor and $,a = \partial / \partial x^a$; L is a scalar density of weight one [1] under an arbitrary coordinate transformation $\bar{x}^i = \bar{x}^i(x^j)$:

$$L = J\left(\frac{\overline{x}}{x}\right)\overline{L} \tag{2}$$

Here we shall employ the property (2) to obtain (in general relativity) in natural manner the energy-momentum pseudotensors of Einstein [2-8], Landau-Lifshitz [9-13], Möller [14-17], Goldberg [18] and Stachel [19], and also the continuity equations of Komar [20], Trautman [21], Du Plessis [22] and Moss [23].

In Sec. 2, we indicate the notation and quantities to employ through the paper. The Sec. 3 is dedicated to the analysis of conservation laws originated from (1), (2) and the Hilbert's variational principle [1, 11, 24, 25]:

$$\delta\int_{V_4} L\sqrt{-g}\,d^4x = 0 \tag{3}$$

We also mention applications of L in the case of general relativity [4-6, 10, 26].

2. Rund-Lovelock's Relations

Rund-Lovelock [1, 24, 25], in their study of variational principles with Lagrangians verifying (1) and (2), showed the importance of the derivatives of L with respect to its arguments:

$$A^{ij} = A^{ji} \equiv \frac{\partial L}{\partial g_{ij}}, \quad A^{ij,h} = A^{ji,h} \equiv \frac{\partial L}{\partial g_{ij,h}}, \quad A^{ij,hk} = A^{ji,hk} = A^{ij,kh} \equiv \frac{\partial L}{\partial g_{ij,hk}}, \tag{4}$$

and, in general, only $A^{ij,hk}$ has tensorial character. These quantities have the following properties:

$$\begin{aligned} A^{ij} &= \frac{L}{2}g^{ij} + \frac{4}{3}A^{hk,jr}R_{k\ \mathrm{hr}}^{\ \mathrm{i}} - \Gamma^{i}{}_{rn}\Gamma^{r}{}_{km}A^{km,rh}, \\ A^{ij,h} &= \Gamma^{i}{}_{rk}A^{rk,jh} + \Gamma^{j}{}_{rk}A^{rk,ih} - \Gamma^{h}{}_{rk}A^{rk,ij}, \\ A^{ij,hk} &= A^{hk,ij}, \ \ A^{ij,hk} + A^{ih,kj} + A^{ik,jh} = 0, \ \ A_{r}{}^{\mathrm{i,jh}}{}_{,\mathrm{hji}} = 0, \end{aligned} \tag{5}$$

with the convention of Dedekind (1868) [27, 28]-Einstein of sum over repeated indices. On the other hand, the Euler-Lagrange expressions defined by:

$$L^{ij} = A^{ij} - A^{ij,h}{}_{,h} + A^{ij,hk}{}_{,hk} \ , \tag{6}$$

can be written using (5), in the form:

$$L^{ij} = \frac{L}{2}g^{ij} + \frac{2}{3}A^{hk,jr}R_{\mathrm{k}\ \mathrm{hr}}^{\ \mathrm{i}} + A^{ij,hk}{}_{;hk} \ , \tag{7}$$

where ;c represents the covariant derivative. Besides, we have the contracted Bianchi identities:

$$L^{ij}{}_{;j} = 0 \ . \tag{8}$$

As an example, if L is the scalar density of weight one corresponding to general relativity [4-6, 10, 26]:

$$L = \sqrt{-g}R, \qquad \mathrm{g} = \det(\mathrm{g}_{\mathrm{ab}}) \quad , \tag{9}$$

where $R \equiv R^{ij}{}_{ji}$ is the scalar curvature, then:

$$A^{ij,km} = \frac{1}{2}\sqrt{-g}\left(2g^{ij}g^{km} - g^{im}g^{jk} - g^{ik}g^{jm}\right) \quad . \tag{10}$$

Thus, (7) and (10) imply that the Euler-Lagrange relations are proportional to the Einstein tensor:

$$L^{ij} = -\sqrt{-g}G^{ij} \quad , \tag{11}$$

which satisfies (8) because $G^{ab}{}_{;b} = 0$.

3. Continuity Equations in Riemannian Spaces

The variational principle (3) is invariant under general transformations $\bar{x}^i = \bar{x}^i(x^j)$, in particular, we can use infinitesimal coordinate changes:

$$\bar{x}^i = x^i + \varepsilon^i \xi^i \left(x^i\right) \quad , \tag{12}$$

without sum over i, and ε^i denoting small constant parameters. The Noether's theorem [29-38] establishes that each continuous symmetry transformation which leaves the corresponding variational principle invariant, implies a conservation law, and hence, a constant of motion. Here we employ the Noether's theorem via the approach of Lanczos [39-42]: we apply the eq. (12) to eq. (3) but now considering that ε^i are new variational variables, then the Lagrange equations for ε^i give the continuity equations of Noether. Thus, it is possible to deduce the following important relations not found explicitly in the literature on gravitational energy-momentum pseudotensors [43-46]:

$$\left(B_r{}^i \xi^r - U_r{}^{ij} \xi^r{}_{,j} - \frac{1}{2} A_r{}^{i,jh} \xi^r{}_{,jh}\right)_{,i} = 0 \quad , \tag{13}$$

where

$$U_r{}^{ij} = A_r{}^{i,jh}{}_{;h} + \Gamma_{krh} A^{kj,ih} - \frac{1}{2}\Gamma^i{}_{hk} A_r{}^{j,hk} \quad , \tag{14}$$

$$B_r{}^i = U_r{}^{ij}{}_{,j} = -\frac{L}{2}\delta^i_r + L_r{}^i + \frac{1}{2}\left(A^{jh,i} - A^{jh,ki}{}_{,k}\right) g_{jh,r} + \frac{1}{2} A^{hk,ij} g_{kh,jr} \quad . \tag{15}$$

Besides, with (8) and (15), it is easy to obtain the conservation law:

$$B_r{}^i{}_{,i} \equiv L_r{}^i{}_{;i} = 0 \quad . \tag{16}$$

In the case of general relativity theory, L given by (9), and a complete study of (13) when ξ^r is a vectorial field, leads to results of Komar [20] and Du Plessis [22], and if ξ^r is a Killing vector, then it is also possible to deduce the relations of Trautman [21] and Moss [23]. On the other hand, (16) allows to construct the energy-momentum pseudotensors of Landau-Lifshitz [9-13], Möller [14-17], Goldberg [18] and Stachel [19].

Sometimes in the Einstein theory, we use the Lagrangian [4, 6, 10, 11]:

$$\bar{L} = \sqrt{-g}\, g^{ab}\left(\Gamma^i{}_{ab}\Gamma^j{}_{ij} - \Gamma^i{}_{ja}\Gamma^j{}_{ib}\right), \tag{17}$$

such that $\sqrt{-g}R = \bar{L} + (ordinary\ divergence)$, then the empty field equations are the same for (9) and (17), besides $\partial\bar{L}/\partial g_{ij,hk} = 0$. However, $\bar{L}$ satisfies (2) when ξ^r are constants, therefore (13) is equivalent to $B_r{}^i{}_{,i} = 0$ and from (15):

$$B_r{}^i = 8\pi\sqrt{-g}\,t_r{}^i = \frac{1}{2}\left(\frac{\partial\bar{L}}{\partial g_{jh,i}} g_{jh,r} - \bar{L}\,\delta^i_r\right), \tag{18}$$

where $t_r{}^i$ is the canonical energy-momentum pseudotensor of Einstein [2-8, 47]. Thus, the conservation law $t_r{}^i{}_{,i} = 0$ is implied by the translational invariance of $\bar{L}$.

4. Conclusions

Therefore, the Lanczos technique [39-42] for the Noether theorem gives the continuity equations (13) and (16) which have total information on energy-momentum quantities for gravitational theories in Riemannian spaces. Thus, the energy-momentum can be regarded as the most fundamental conserved quantity being associated with symmetry of the space time geometry.

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