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Research Article

Some Inequalities of Certain Subclass of Meromorphic Functions Defined by Using New Integral Operator

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Abstract: By using new integral operator, we investigate two interesting properties of certain class of p-valent meromorphic functions. Moreover we obtained the corresponding results due to Liu-Srivastava operator and Uralegaddi-Somanatha operator, which are the meromorphic analogous of Carlson-Shaffer operator and Ruscheweyh operator, respectively.

Keywords: Meromorphic functions; multivalent functions; differential subordination.

1 Introduction

Let $\Sigma_{p,n}$ denote the class of multivalent meromorphic functions of the form:

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, ...\}\,; n > -p), \tag{1.1}$$

which are analytic in the punctured unit disc $U^* = U \setminus \{0\}\,; U = \{z \in \mathbb{C} : |z| < 1\}$. For two functions $f(z)$ and $g(z)$, analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ which (by definition) is analytic in U, satisfying the following conditions:

$$w(0) = 0 \text{ and } |w(z)| < 1; \;\; (z \in U)$$

such that

$$f(z) = g(w(z)); \;\; (z \in U)\,,$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in U) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

In particular, If the function $g(z)$ is univalent in U, we have the following equivalence (see [3], [7] and [8]):

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\varphi(r, s; z) : \mathbb{C}^2 \times U \longrightarrow \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\varphi\left(p(z), zp'(z); z\right) \prec h(z) \tag{1.2}$$

then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\widetilde{q}$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants of (1.2) is called the best dominant (see [8]).

Following the recent work of El-Ashwah [4], for $\mu > 0$, $a,c \in \mathbb{C}$ be such that $Re(c - a) \geq 0$, $Re(a) \geq \mu p$ $(p \in \mathbb{N})$ and $f(z) \in \Sigma_{p,n}$ given by (1.1), we define the integral operator

$$J_{p,\mu}^{a,c} : \Sigma_{p,n} \longrightarrow \Sigma_{p,n}$$

as following:

(i) for $Re(c - a) > 0$ by:

$$J_{p,\mu}^{a,c} f(z) = \frac{\Gamma(c - \mu p)}{\Gamma(a - \mu p)\Gamma(c - a)} \int_0^1 t^{a-1}(1 - t)^{c-a-1} f(zt^{\mu})dt; \tag{1.3}$$

(ii) for $a = c$ by:

$$J_{p,\mu}^{a,a} f(z) = f(z). \tag{1.4}$$

Using (1.3) and (1.4), it is easily to deduce that the operator $J_{p,\mu}^{a,c} f(z)$ can be expressed as following:

$$J_{p,\mu}^{a,c} f(z) = z^{-p} + \frac{\Gamma(c - \mu p)}{\Gamma(a - \mu p)} \sum_{k=n}^{\infty} \frac{\Gamma(a + \mu k)}{\Gamma(c + \mu k)} a_k z^k, \tag{1.5}$$

$$\left(\mu > 0; a, c \in \mathbb{C}, Re(c - a) \geq 0; Re(a) > p\mu; p \in \mathbb{N}; n > -p\right).$$

Using (1.5), we can obtain the following recurrence relation, which is needed for our investigations:

$$z\left(J_{p,\mu}^{a,c+1} f(z)\right)' = \frac{c - p\mu}{\mu} J_{p,\mu}^{a,c} f(z) - \frac{c}{\mu} J_{p,\mu}^{a,c+1} f(z). \tag{1.6}$$

By specializing the parameters in (1.5), we note that:

(i) $J_{p,1}^{a+p,c+p} f(z) = \ell_p(a,c)$ $\left(a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0,1,2,...\}; p \in \mathbb{N}\right)$ (see Liu and Srivastava [6]);

(ii) $J_{p,1}^{n+2p,p+1} f(z) = D^{n+p-1} f(z)$ (n is an integer, $n > -p$, $p \in \mathbb{N}$) (see Uralegaddi and Somanatha [9], Aouf [1] and Aouf and Srivastava [2]).

Thus, the new results obtained in this paper can give us the corresponding results of the other well-known operators by some special choices of the parameters a, c and μ.

2 Main Results

Unless otherwise mentioned we shall assume throughout the recent section that $\mu > 0$, $a, c \in \mathbb{R}$, $c \geq a$, $a > \mu p$, $p \in \mathbb{N}$ and $n > -p$.
In order to introduce our results, we need the following lemma.

Lemma 2.1 (5) *Let $h(z)$ be analytic and convex (univalent) in U, $h(0) = 1$, and let*

$$\varphi(z) = 1 + c_{p+n} z^{p+n} + ... \tag{2.1}$$

be analytic in U. If

$$\varphi(z) + \frac{1}{\delta} z\varphi'(z) \prec h(z),$$

then for $\delta \neq 0$ and $Re(\delta) \geq 0$, we have

$$\varphi(z) \prec \psi(z) = \left(\frac{\delta}{p+n}\right) z^{-\left(\frac{\delta}{p+n}\right)} \int_0^z t^{\left(\frac{\delta}{p+n}\right)-1} h(t)dt, \quad (z \in U) \tag{2.2}$$

and $\psi(z)$ is the best dominant of (2.2).

Theorem 2.2 *For $-1 \leq B < A \leq 1$, $0 < \gamma < 1$ and $f(z) \in \Sigma_{p,n}$. Suppose that*

$$\sum_{k=n}^{\infty} c_k |a_k| \leq 1, \tag{2.3}$$

where

$$c_k = \frac{1-B}{A-B}\left[(c+\mu k) - \mu\gamma(k+p)\right] \frac{\Gamma(c-\mu p)\Gamma(a+\mu k)}{\Gamma(a-\mu p)\Gamma(c+1+\mu k)}. \tag{2.4}$$

Then we have the following:
(i) If $-1 \leq B \leq 0$, then

$$(1-\gamma) z^p J_{p,\mu}^{a,c} f(z) + \gamma z^p J_{p,\mu}^{a,c+1} f(z) \prec \frac{1+Az}{1+Bz}. \tag{2.5}$$

(ii) If $-1 \leq B \leq 0$ and $\rho \geq 1$, then

$$Re\left\{\left(z^p J_{p,\mu}^{a,c+1} f(z)\right)^{\frac{1}{\rho}}\right\} > \left\{\frac{c-\mu p}{\mu\left(1-\gamma\right)\left(p+n\right)} \int\limits_0^1 t^{\left(\frac{c-\mu p}{\mu(1-\gamma)(p+n)}\right)-1} \frac{1-At}{1-Bt} dt\right\}^{\frac{1}{\rho}}. \tag{2.6}$$

The result is sharp.

Proof: We begin with proving the first assertion of the theorem which is the subordination property (2.5). Let

$$G(z) = \left(1-\gamma\right) z^p J_{p,\mu}^{a,c} f(z) + \gamma z^p J_{p,\mu}^{a,c+1} f(z), \tag{2.7}$$

then making use of (1.5), we obtain

$$G(z) = 1 + \sum_{k=n}^{\infty} \Big[\left(c+\mu k\right) - \mu\gamma\left(k+p\right)\Big] \frac{\Gamma\left(c-\mu p\right)\Gamma\left(a+\mu k\right)}{\Gamma\left(a-\mu p\right)\Gamma\left(c+1+\mu k\right)} a_k z^{k+p}. \tag{2.8}$$

Also, using (2.3) and the assumption that $-1 \leq B \leq 0$, then for $z \in U$, we obtain

$$\begin{aligned}
\left|\frac{G(z)-1}{A-BG(z)}\right| &= \left|\frac{\sum\limits_{k=n}^{\infty} \Big[\left(c+\mu k\right)-\mu\gamma\left(k+p\right)\Big] \frac{\Gamma(c-\mu p)\Gamma(a+\mu k)}{\Gamma(a-\mu p)\Gamma(c+1+\mu k)} a_k z^{k+p}}{A-B-B\sum\limits_{k=n}^{\infty} \Big[\left(c+\mu k\right)-\mu\gamma\left(k+p\right)\Big] \frac{\Gamma(c-\mu p)\Gamma(a+\mu k)}{\Gamma(a-\mu p)\Gamma(c+1+\mu k)} a_k z^{k+p}}\right| \\
&\leq \frac{\sum\limits_{k=n}^{\infty} c_k \left|a_k\right|}{A-B+B\sum\limits_{k=n}^{\infty} c_k \left|a_k\right|} \\
&\leq 1,
\end{aligned}$$

which proves (2.5) of the theorem.
To prove the second assertion of the theorem, let

$$\varphi(z) = z^p J_{p,\mu}^{a,c+1} f(z), \tag{2.9}$$

Then the function $\varphi(z)$ takes the form (2.1) and analytic in U. Differentiating (2.9) with respect to z and using (1.8), we obtain

$$\varphi(z) + \frac{\mu\left(1-\gamma\right)}{\left(c-\mu p\right)} z\varphi'(z) = \left(1-\gamma\right) z^p J_{p,\mu}^{a,c} f(z) + \gamma z^p J_{p,\mu}^{a,c+1} f(z) \prec \frac{1+Az}{1+Bz}, \tag{2.10}$$

using Lemma 1, then we obtain

$$\varphi(z) \prec \frac{c-\mu p}{\mu\left(1-\gamma\right)\left(p+n\right)} z^{-\left(\frac{c-\mu p}{\mu(1-\gamma)(p+n)}\right)} \int\limits_0^z t^{\left(\frac{c-\mu p}{\mu(1-\gamma)(p+n)}\right)-1} \frac{1+At}{1+Bt} dt,$$

which is equivalent to the following

$$z^p J_{p,\mu}^{a,c+1} f(z) = \frac{c-\mu p}{\mu\left(1-\gamma\right)\left(p+n\right)} \int\limits_0^1 u^{\left(\frac{c-\mu p}{\mu(1-\gamma)(p+n)}\right)-1} \frac{1+Auw(z)}{1+Buw(z)} du. \tag{2.11}$$

Where $w(z)$ is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$.
Moreover, from (2.11) we have

$$Re\left\{z^{p}J_{p,\mu}^{a,c+1}f(z)\right\}>\frac{c-\mu p}{\mu\left(1-\gamma\right)\left(p+n\right)}\int_{0}^{1}u^{\left(\frac{c-\mu p}{\mu(1-\gamma)(p+n)}\right)-1}\ \frac{1-Au}{1-Bu}du>0\ \ (z\in U). \tag{2.12}$$

Now, in (2.12) apply the elementary inequality $Re\left\{w^{\frac{1}{\rho}}\right\}\geq\left(Re\left\{w\right\}\right)^{\frac{1}{\rho}}$ for $Re\left\{w\right\}>0$ $(\rho\in\mathbb{N})$, then inequality (2.6) follows immediately.
To show the sharpness of (2.6), take $f(z)\in\Sigma_{p,n}$ defined by

$$z^{p}J_{p,\mu}^{a,c+1}f(z)=\frac{c-\mu p}{\mu\left(1-\gamma\right)\left(p+n\right)}\int_{0}^{1}u^{\left(\frac{c-\mu p}{\mu(1-\gamma)(p+n)}\right)-1}\ \frac{1+Auz^{n}}{1+Buz^{n}}du. \tag{2.13}$$

For this function we find that

$$\left(1-\gamma\right)z^{p}J_{p,\mu}^{a,c}f(z)+\gamma z^{p}J_{p,\mu}^{a,c+1}f(z)=\frac{1+Az^{n}}{1+Bz^{n}},$$

and

$$z^{p}J_{p,\mu}^{a,c+1}f(z)\longrightarrow\frac{c-\mu p}{\mu\left(1-\gamma\right)\left(p+n\right)}\int_{0}^{1}u^{\left(\frac{c-\mu p}{\mu(1-\gamma)(p+n)}\right)-1}\ \frac{1-Au}{1-Bu}du\ \text{ as }\ z\longrightarrow e^{\frac{i\pi}{n}}.$$

Hence the proof of Theorem 1 is completed.

Taking $p=1$ in Theorems 1, we obtain the following corollary.

Corollary 2.3 *For $-1\leq B<A\leq 1$, $0<\gamma<1$ and $f(z)\in\Sigma_{p,n}$. Suppose that*

$$\sum_{k=n}^{\infty}c_{k}\left|a_{k}\right|\leq 1,$$

where

$$c_{k}=\frac{1-B}{A-B}\Big[\left(c+\mu k\right)-\mu\gamma\left(k+1\right)\Big]\frac{\Gamma\left(c-\mu\right)\Gamma\left(a+\mu k\right)}{\Gamma\left(a-\mu\right)\Gamma\left(c+1+\mu k\right)}. \tag{2.14}$$

Then we have the following:
(i) If $-1\leq B\leq 0$, then

$$\left(1-\gamma\right)zI_{\mu}^{a,c}f(z)+\gamma zI_{\mu}^{a,c+1}f(z)\prec\frac{1+Az}{1+Bz}. \tag{2.15}$$

(ii) If $-1\leq B\leq 0$ and $\rho\geq 1$, then

$$Re\left\{\left(zI_{\mu}^{a,c+1}f(z)\right)^{\frac{1}{\rho}}\right\}>\left\{\frac{c-\mu}{\mu\left(1-\gamma\right)\left(1+n\right)}\int_{0}^{1}t^{\left(\frac{c-\mu}{\mu(1-\gamma)(1+n)}\right)-1}\ \frac{1-At}{1-Bt}dt\right\}^{\frac{1}{\rho}}. \tag{2.16}$$

The result is sharp.

Taking $a = a + p\,(a \in \mathbb{R})$, $c = c + p\left(c \in \mathbb{R}\backslash\mathbb{Z}_0^-\right)$ and $\mu = 1$ in Theorems 1, we obtain the following corollary.

Corollary 2.4 *For $-1 \leq B < A \leq 1$, $0 < \gamma < 1$ and $f(z) \in \Sigma_{p,n}$. Suppose that*

$$\sum_{k=n}^{\infty} c_k\,|a_k| \leq 1,$$

where

$$c_k = \frac{1-B}{A-B}\Big[(c+p+k) - \gamma\,(k+p)\Big]\frac{\Gamma\,(c)\,\Gamma\,(a+p+k)}{\Gamma\,(a)\,\Gamma\,(c+p+k+1)}. \tag{2.17}$$

Then we have the following:
(i) If $-1 \leq B \leq 0$, then

$$(1-\gamma)\,z^p\ell_p(a,c)f(z) + \gamma z^p\ell_p(a,c+1)f(z) \prec \frac{1+Az}{1+Bz}. \tag{2.18}$$

(ii) If $-1 \leq B \leq 0$ and $\rho \geq 1$, then

$$Re\left\{\left(z^p\ell_p(a,c+1)f(z)\right)^{\frac{1}{\rho}}\right\} > \left\{\frac{c}{(1-\gamma)\,(p+n)}\int_0^1 t^{\left(\frac{c}{(1-\gamma)(p+n)}\right)-1}\;\frac{1-At}{1-Bt}dt\right\}^{\frac{1}{\rho}}. \tag{2.19}$$

The result is sharp.

Taking $a = n + 2p\,(n > -p,\ p \in \mathbb{N})$, $c = p + 1\,(p \in \mathbb{N})$ and $\mu = 1$ in Theorems 1, we obtain the following corollary.

Corollary 2.5 *For $-1 \leq B < A \leq 1$, $0 < \gamma < 1$ and $f(z) \in \Sigma_{p,n}$. Suppose that*

$$\sum_{k=n}^{\infty} c_k\,|a_k| \leq 1,$$

where

$$c_k = \frac{1-B}{A-B}\Big[(k+p)\,(1-\gamma) + 1\Big]\frac{\Gamma\,(n+2p+k)}{\Gamma\,(n+p)\,\Gamma\,(p+2+k)}. \tag{2.20}$$

Then we have the following:
(i) If $-1 \leq B \leq 0$, then

$$(1-\gamma)\,z^pD^{n+p-1}f(z) + \gamma z^pD^{n+p}f(z) \prec \frac{1+Az}{1+Bz}. \tag{2.21}$$

(ii) If $-1 \le B \le 0$ and $\rho \ge 1$, then

$$Re\left\{\left(z^p D^{n+p} f(z)\right)^{\frac{1}{\rho}}\right\} > \left\{\frac{1}{(1-\gamma)(p+n)}\int_0^1 t^{\left(\frac{1}{(1-\gamma)(p+n)}\right)-1}\ \frac{1-At}{1-Bt}dt\right\}^{\frac{1}{\rho}}. \tag{2.22}$$

The result is sharp.

Theorem 2.6 *Let $f(z) \in \Sigma_{p,n}$ be given by (1.1) and let*

$$c_k \ge \begin{cases} 1; & k = n, n+1, ..., q; \\ c_{q+1}; & k = q+1, q+2, ... \ , \end{cases}$$

where c_k is given by (2.4) and satisfying the condition (2.3), define the partial sums $s_1(z)$ and $s_q(z)$ by

$$s_1(z) = z^{-p} \quad and \ s_q(z) = z^{-p} + \sum_{k=n}^{q} a_k z^k \quad (q \in \mathbb{N}; q > n). \tag{2.23}$$

Then

$$Re\left\{\frac{f(z)}{s_q(z)}\right\} > 1 - \frac{1}{c_{q+1}} \ (z \in U; q \in \mathbb{N}; q > n), \tag{2.24}$$

and

$$Re\left\{\frac{s_q(z)}{f(z)}\right\} > 1 - \frac{1}{1+c_{q+1}} \ (z \in U; q \in \mathbb{N}; q > n). \tag{2.25}$$

The estimates in (2.24) and (2.25) are sharp for $q \in \mathbb{N}$ and $q > n$.

Proof: (i) Under the hypothesis of Theorem 1, we can see from (2.3) that

$$\sum_{k=n}^{q} |a_k| + c_{q+1} \sum_{k=q+1}^{\infty} |a_k| \le \sum_{k=n}^{\infty} c_k |a_k| \le 1. \tag{2.26}$$

By setting

$$g_1(z) = c_{q+1}\left\{\frac{f(z)}{s_q(z)} - \left(1 - \frac{1}{c_{q+1}}\right)\right\} = 1 + \frac{c_{q+1}\sum\limits_{k=q+1}^{\infty} a_k z^{k+p}}{1 + \sum\limits_{k=n}^{q} a_k z^{k+p}}, \tag{2.27}$$

and applying (2.26), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{c_{q+1}\sum\limits_{k=q+1}^{\infty} |a_k|}{2 - 2\sum\limits_{k=n}^{q} |a_k| - c_{q+1}\sum\limits_{k=q+1}^{\infty} |a_k|} \le 1 \ (z \in U), \tag{2.28}$$

which readily yields the assertion (2.24) of Theorem 2. If we take

$$f(z) = z^{-p} + \frac{z^{q+1}}{c_{q+1}} \ (q \in \mathbb{N}; q > n), \tag{2.29}$$

with $z = re^{\frac{i\pi}{q+p+1}}$ and let $r \longrightarrow 1^-$, we obtain

$$\frac{f(z)}{s_q(z)} = 1 + \frac{z^{q+p+1}}{c_{q+1}} \longrightarrow 1 - \frac{1}{c_{q+1}},$$

which shows that the bound in (2.24) is best possible for each $q \in \mathbb{N}$ and $q > n$.
(ii) Similarly, if we put

$$g_2(z) = (1 + c_{q+1}) \left\{ \frac{s_q(z)}{f(z)} - \frac{c_{q+1}}{1 + c_{q+1}} \right\} = 1 - \frac{(1 + c_{q+1}) \sum\limits_{k=q+1}^{\infty} |a_k| \, z^{k+p}}{1 + \sum\limits_{k=n}^{\infty} |a_k| \, z^{k+p}},$$

and make use of (2.26), we can deduce that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + c_{q+1}) \sum\limits_{k=q+1}^{\infty} |a_k|}{2 - 2 \sum\limits_{k=n}^{q} |a_k| + (1 - c_{q+1}) \sum\limits_{k=q+1}^{\infty} |a_k|} \leq 1 \ \ (z \in U), \tag{2.30}$$

which yields inequality (2.25) of Theorem 2. The bound in (2.15) is sharp for each $q \in \mathbb{N}$ and $q > n$, with the extreme function $f(z)$ given by (2.29). The proof of Theorem 2 is now completed.

Theorem 2.7 *Let $f(z) \in \Sigma_{p,n}$ be given by (1.1) and let*

$$c_k \geq \begin{cases} \frac{k}{p}; & k = n, n+1, ..., q; \\ \frac{c_{q+1}}{q+1} \left(\frac{k}{p} \right); & k = q+1, q+2, \dots \ , \end{cases}$$

where c_k is given by (2.4) and satisfying the condition (2.3), then we have

$$Re \left\{ \frac{f'(z)}{s'_q(z)} \right\} > 1 - \frac{q+1}{c_{q+1}} \ (q \in \mathbb{N}; q > n; z \in U), \tag{2.31}$$

and

$$Re \left\{ \frac{s'_q(z)}{f'(z)} \right\} > 1 - \frac{q+1}{q+1+c_{q+1}} \ (q \in \mathbb{N}; q > n; z \in U). \tag{2.32}$$

The estimates in (2.31) and (2.32) are sharp for $f(z)$ defined by (2.29).

Proof: By setting

$$g_3(z) = \frac{c_{q+1}}{q+1}\left\{\frac{f'(z)}{s_q'(z)} - \left(1 - \frac{q+1}{c_{q+1}}\right)\right\} = 1 + \frac{\frac{c_{q+1}}{q+1}\sum\limits_{k=q+1}^{\infty}\frac{k}{p}a_k z^{k+p}}{-1 + \sum\limits_{k=n}^{q}\frac{k}{p}a_k z^{k+p}}, \tag{2.33}$$

Then, find that

$$\left|\frac{g_3(z)-1}{g_3(z)+1}\right| \leq \frac{\frac{c_{q+1}}{q+1}\sum\limits_{k=q+1}^{\infty}\frac{k}{p}|a_k|}{2 - 2\sum\limits_{k=n}^{q}\frac{k}{p}|a_k| - \frac{c_{q+1}}{q+1}\sum\limits_{k=q+1}^{\infty}\frac{k}{p}|a_k|}. \tag{2.34}$$

Now,

$$\left|\frac{g_3(z)-1}{g_3(z)+1}\right| \leq 1,$$

since

$$\sum_{k=n}^{q}\frac{k}{p}|a_k| + \frac{c_{q+1}}{q+1}\sum_{k=q+1}^{\infty}\frac{k}{p}|a_k| \leq \sum_{k=n}^{\infty}c_k|a_k| \leq 1. \tag{2.35}$$

Then, the proof of (2.31) is completed.
To prove (2.32), we define the function $g_4(z)$ by

$$\begin{aligned} g_4(z) &= \left(\frac{q+1+c_{q+1}}{q+1}\right)\left\{\frac{s_q'(z)}{f'(z)} - \frac{c_{q+1}}{q+1+c_{q+1}}\right\} \\ &= 1 - \frac{\left(1+\frac{c_{q+1}}{q+1}\right)\sum\limits_{k=q+1}^{\infty}\frac{k}{p}a_k z^{k+p}}{-1 + \sum\limits_{k=n}^{\infty}\frac{k}{p}a_k z^{k+p}}, \end{aligned}$$

and making use of (2.35), we deduce that

$$\left|\frac{g_4(z)-1}{g_4(z)+1}\right| \leq \frac{\left(1+\frac{c_{q+1}}{q+1}\right)\sum\limits_{k=q+1}^{\infty}\frac{k}{p}|a_k|}{2 - 2\sum\limits_{k=n}^{\infty}\frac{k}{p}|a_k| - \left(1+\frac{c_{q+1}}{q+1}\right)\sum\limits_{k=q+1}^{\infty}\frac{k}{p}|a_k|} \leq 1,$$

which leads us immediately to the assertion (2.32) of Theorem 3.

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