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Research Article

On an Iterative Method to Solve 2th Order Homogeneous Linear Differential Equations

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Abstract: The works [1-5] employ an asymptotic iteration method to solve 2th order homogeneous linear differential equations, searching a special structure for one of the corresponding solutions. In such References is not mentioned that this type of solution must be polynomial and that its existence is governed by a Riccati's equation, which is evident in the present approach.

Keywords: Riccati's equation; Asymptotic iteration method.

1. Introduction

The method exhibited in [1-5] concerns to the homogeneous differential equation:

$$y'' - \lambda_0(x)y' - S_0(x)y = 0, \;\; \lambda_0 \neq 0, \tag{1}$$

searching a solution with the structure:

$$y = \exp\left(-\int^x \alpha(t)dt\right), \tag{2}$$

then we must indicate the conditions for the existence of $\alpha(x)$ such that (2) verifies (1), which is realized in Sec. 2, thus α satisfies the Riccati equation [6-9] and the procedure from [1-5] only gives polynomial solutions for (1). In Sec. 3 we make applications to Hermite, Chebyshev and Laguerre equations.

2. Iterative method

The proposal (2) implies:

$$y' = -\alpha y, \tag{3.a}$$

$$y'' = (\alpha^2 - \alpha')y, \tag{3.b}$$

then (1) gives the compatibility condition:

$$\alpha' - \alpha^2 - \lambda_0\alpha + S_0 = 0, \tag{4}$$

a Riccati equation [6-9], whose importance is not explicitly in [1, 3-5] but it is mentioned in [2]. That is, $\alpha(x)$ must satisfy (4) if (2) is a solution of (1).

In (3.b) we employ (4) to obtain:

$$y'' = (S_0 - \lambda_0\alpha)y, \tag{5}$$

thus it is natural to search the cancellation of the right member:

$$\alpha = \frac{S_0}{\lambda_0}, \tag{6}$$

$$\therefore \quad y'' = 0, \tag{7.a}$$

which implies:

$$y''' = 0, \tag{7.b}$$

then it is necessary to apply $\frac{d}{dx}$ to (1) and to use (5) to eliminate y'' in the final expression:

$$y''' - \lambda_1 y' - S_1 y = 0, \tag{8.a}$$

$$S_1 = S_0' + S_0\lambda_0 \ , \ \lambda_1 = \lambda_0' + \lambda_0^2 + S_0 \ . \tag{8.b}$$

If (3.a) is applied in (8.a):

$$y''' = (S_1 - \lambda_1\alpha)y,$$

therefore from (6) and (7.b) we deduce the restriction:

$$\alpha = \frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1}, \tag{9}$$

where we substitute the relations (8.b) to obtain the compatibility condition:

$$R_0 \equiv \lambda_0 S_0' - S_0\lambda_0' - S_0^2 = 0, \tag{10}$$

The proposal (6) gives the solution (2) if $\alpha = {S_0}/{\lambda_0}$ satisfies (4), in fact:

$$\frac{d}{dx}\left(\frac{S_0}{\lambda_0}\right) - \left(\frac{S_0}{\lambda_0}\right)^2 - S_0 + S_0 = 0,$$

with $R_0 = 0$, in harmony with (10). In resumé, (2) is a solution of (1) with α given by (6) if the functions S_0 and λ_0 verify (10) [which it is equivalent to ask (9)].

From (4) and (6) we deduce that $\alpha^{'} - \alpha^2 = 0$, whose solution is immediate:

$$\alpha = \frac{S_0}{\lambda_0} = \frac{1}{q_1 - x}, \tag{11.a}$$

being q_1 an arbitrary constant, thus (2) implies:

$$y(x) = q_1 - x, \tag{11.b}$$

then $y^{'} = -1$ & $y^{''} = 0$, in accordance with (3.a), (6), (7.a) and (11.a). It is important to emphasize that in the case (9) the functions S_0 and λ_0 must satisfy (11.a), and thus necessarily the solution of (1) shall have the form (11.b). The equation (1) is homogeneous, therefore any constant multiple of (11.b) also is a solution.

From (7.b), $y^{'''} = 0$, then:

$$y^{iv} = 0 \tag{12}$$

which originates the application of $\frac{d}{dx}$ to (8.a), where we employ (1) to eliminate y'' in the final relation:

$$y^{iv} - \lambda_2 y^{'} - S_2 y = 0, \tag{13.a}$$

$$S_2 = S_1^{'} + S_0\lambda_1 , \qquad \lambda_2 = \lambda_1^{'} + \lambda_0\lambda_1 + S_1 \tag{13.b}$$

If in (13.a) we use (3.a):

$$y^{iv} = (S_2 - \lambda_2\alpha)y \ ,$$

and its comparison with (12) implies:

$$\alpha = \frac{S_2}{\lambda_2} = \frac{S_0}{\lambda_0}, \tag{14}$$

of easy verification because there we can substitute (13.b) to deduce that:

$$\frac{d}{dx}R_0 + \lambda_0 R_0 = 0 \ \therefore 0 = 0 \qquad \text{[from (10)]}.$$

With successive derivatives we can construct similar equations to (8.a), (13.a), etc., therefore:

$$y^{(n+2)} - \lambda_n y^{'} - S_n y = 0, \tag{15.a}$$

$$S_n = S_{n-1}^{'} + S_0\lambda_{n-1}, \qquad \lambda_n = \lambda_{n-1}^{'} + \lambda_0\lambda_{n-1} + S_{n-1}, \tag{15.b}$$

$$y^{(n+2)} = (S_n - \lambda_n\alpha)y, \tag{15.c}$$

and we notice that:

$$\alpha = \frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} \quad => \quad \alpha = \frac{S_2}{\lambda_2} = \frac{S_3}{\lambda_3} = \cdots,$$

because $y^{(m)} = 0$, m=2,3…

Instead (9), it may occur that:

$$\alpha = \frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2}, \tag{16.a}$$

which it would imply:

$$\alpha = \frac{S_3}{\lambda_3} = \frac{S_4}{\lambda_4} = \cdots, \tag{16.b}$$

because $y^{(r)} = 0, r = 3,4, \ldots$, with the fulfillment of the Riccati equation (4) and $y(x)$ an polynomial of order two:

$$y(x) = q_2 + q_1 x - x^2 \ , \tag{17.a}$$

where the constants q_1 and q_2 are obtained via:

$$\alpha = -\frac{q_1 - 2x}{q_2 + q_1 x - x^2}, \tag{17.b}$$

and from (5) and (17.a) we deduce the relation:

$$q_2 S_0 + q_1(\lambda_0 + S_0 x) - (2\lambda_0 + S_0 x)x = -2, \tag{18}$$

instead (11.a). We remember that S_0 and λ_0 are data; if we find constants q_1 and q_2 verifying (18) then (17.a) is a solution of (1), besides also the expressions (16.a, b) and (17.b) are satisfied.

The previous results can be generalized, in fact, if we have *n* such that:

$$\alpha = \frac{S_{n-1}}{\lambda_{n-1}} = \frac{S_n}{\lambda_n}, \tag{19}$$

then $y(x)$ given by (2) is a polynomial of degree *n*:

$$y(x) = q_n + q_{n-1}x + \cdots + q_1 x^{n-1} - x^n \ , \tag{20.a}$$

and it is a solution of (1), where the constants q_j are constructed via:

$$\alpha = -\frac{y'}{y} = -\frac{q_{n-1} + 2q_{n-2}x + \cdots + (n-1)q_1 x^{n-2} - n x^{n-1}}{q_n + q_{n-1}x + \cdots + q_1 x^{n-1} - x^n} \tag{20.b}$$

With (15.b) and (19) we can to verify the fulfillment of (4), which permits to guarantee that (2) is solution of (1). The condition (19) implies:

$$\alpha = \frac{S_{n+1}}{\lambda_{n+1}} = \frac{S_{n+2}}{\lambda_{n+2}} = \cdots,$$

because $y^{(m)} = 0$, $m = n + 1, n + 2, \ldots$

We emphasize that all solutions of (1), with the form (2) and the property (19), are polynomials, which is not indicated explicitly in [1, 3-5], besides in these References does not mentioned the Riccati equation (4) which is fundamental for the existence of a solution with the characteristics (2) and (19), to see [2].

It is known [10] that if $y_1(x)$ is a solution of the homogeneous equation:

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{21.a}$$

then the other solution of (21.a) is given by:

$$y_2(x) = y_1(x) \int^x \frac{\exp\left(-\int^\eta \frac{q(t)}{p(t)}\, dt\right)}{[y_1(\eta)]^2} d\eta \tag{21.b}$$

The comparison of (1) and (21.a) gives $p = 1$, $q = -\lambda_0$, $r = -S_0$ & $y_1 = \exp\left(-\int^x \alpha(t)\, dt\right)$, thus from (21.b) results:

$$y_2 = y_1 \int^x \exp\left[\int^\eta \left(\lambda_0(t) + 2\alpha(t)\right) dt\right] d\eta\,, \tag{22}$$

which is the equation (2.12) in [1] (or the equation (8) in [2]). We indicate that $y_1(x)$ always is a polynomial due to (19), however, $y_2(x)$ is not necessarily a polynomial.

3. Hermite, Chebyshev and Laguerre Equations

Here we apply the iterative method exhibited in Sec. 2 to construct polynomial solutions.

a).- Hermite's equation [11-14]:

$$y'' - 2xy' + 2ky = 0,$$

with $k = 1$, then $S_0 = -2, \lambda_0 = 2x,$ and from (8.b) results that $S_1 = -4x,\ \lambda_1 = 4x^2$, besides (9) is verified:

$$\alpha = \frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} = -\frac{1}{x} \quad ,$$

and its comparison with (11.a) gives $q_1 = 0$, and (11.b) generates the corresponding solution of (1):

$$y(x) = x \propto H_1(x) = 2x.$$

If now we employ $k = 2$, we have $S_0 = -4,\ \lambda_0 = 2x,\ S_1 = -8x,\ \lambda_1 = 4x^2 - 2,\ S_2 = -16x^2,\ \lambda_2 = 8x^3 - 4x,$ where we used (13.b), and (16.a) is satisfied:

$$\alpha = \frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2} = -\frac{-2x}{\frac{1}{2} - x^2} \quad ,$$

with the structure (17.b), thus $q_1 = 0,\ q_2 = \frac{1}{2}$, and the solution (17.a) adopts the form:

$$y(x) = \frac{1}{2} - x^2 \propto H_2(x) = -2 + 4x^2, \qquad etc.$$

In this manner, for each value of *k* we can to construct the corresponding Hermite's polynomial.

b).- Chebyshev's equation [12, 15, 16]:

$$y'' - \frac{x}{1 - x^2} y' + \frac{k^2}{1 - x^2} y = 0 \ ,$$

for $k = 3$, then:

$$S_0 = -\frac{9}{1 - x^2} \ , \qquad \lambda_0 = \frac{x}{1 - x^2} \ , \qquad S_1 = -\frac{27x}{(1 - x^2)^2} \ ,$$

$$\lambda_1 = \frac{11x^2 - 8}{(1 - x^2)^2} \ , \qquad S_2 = \frac{45(1 - 4x^2)}{(1 - x^2)^3} \ , \qquad \lambda_2 = \frac{15(4x^3 - 3x)}{(1 - x^2)^3} \ , \ldots$$

such that:

$$\alpha = \frac{S_2}{\lambda_2} = \frac{S_3}{\lambda_3} = -\frac{\frac{3}{4} - 3x^2}{\frac{3}{4}x - x^3} \quad ,$$

with $q_1 = 0,\ q_2 = \frac{3}{4},\ q_3 = 0$, and from (20.a):

$$y(x) = \frac{3}{4}x - x^3 \propto T_3(x) = -3x + 4x^3, \qquad etc,$$

Thus, for each value of *k*, this method gives a Chebyshev's polynomial.

c).- Laguerre's equation [12, 13, 17, 18]:

$$y'' + \frac{1 - x}{x} y' + \frac{k}{x} y = 0 \ ,$$

with *k*=3, therefore:

$$S_0 = -\frac{3}{x} \ , \qquad \lambda_0 = -\frac{1 - x}{x} \ , \qquad S_1 = \frac{6 - 3x}{x^2} \ , \qquad \lambda_1 = \frac{x^2 - 5x + 2}{x^2} \ ,$$

$$S_2 = \frac{-18 + 18x - 3x^2}{x^3} \ , \qquad \lambda_2 = \frac{-6 + 18x - 9x^2 + x^3}{x^3} \ , \ldots$$

and (20.b) is satisfied:

$$\alpha = \frac{S_2}{\lambda_2} = \frac{S_3}{\lambda_3} = -\frac{-18+18x-3x^2}{6-18x+9x^2-x^3} \ ,$$

for $q_1 = 9$, $q_2 = -18$, $q_3 = 6$:

$$y(x) = 6 - 18x + 9x^2 - x^3 = 6L_3(x) \ , \qquad etc.$$

We conclude that this iterative method is useful to construct polynomial solutions of differential equations of second order, linear and homogeneous.

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