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Research Article

On a Subclass of Univalent Functions With Fixed Second Coefficient Defined by Generalized Derivative Operator

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Abstract: In this paper we have introduced the subclass of univalent functions defined in the open unit disc and derived some interesting properties like coefficient inequality, distortion theorem, extreme points, radii of starlikeness and convexity and integral mean inequalities.

Keywords: Uniform convex function; Convex function; Generalized derivative operator; Distortion bounds.

1 Introduction

Let $\mathcal{U}$ denote the open unit disc,

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}, \tag{1}$$

and let S denote the class of analytic and univalent functions in $\mathcal{U}$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

Goodman [3,4] introduced and defined the classes of uniformly convex UCV and uniformly starlike UST functions in $\mathcal{U}$. We have from [3], [7] that

$$f \in UCV \leftrightarrow \left| \frac{zf''(z)}{f'(z)} \right| \leq \Re \left\{ \frac{zf''(z)}{f'(z)} \right\},$$

and

$$f \in S_p \leftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left\{ \frac{zf'(z)}{f(z)} \right\},$$

where S_p is a new class of starlike functions introduced by Rϕnning [7].
For any two function f and g analytic in $\mathcal{U}$, f is said to be subordinate to g in $\mathcal{U}$, denoted by $f \prec g$ if there exists an analytic function ω defined in $\mathcal{U}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z)), z \in \mathcal{U}$.
In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$[6]. For $f \in S$ we introduce a generalized derivative operator for f given by Eljamal and Darus in [2], for fixed positive natural number m and $\lambda_2 \geq \lambda_1 \geq 0$.

$$D^{m,k}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} (\frac{1 + (\lambda_1 + \lambda_2)(k-1)}{1 + \lambda_2(k-1)})^m a_k z^k.$$

Note that the following are the special cases of $D^{m,k}_{\lambda_1,\lambda_2}$:

1. when $\lambda_1 = 1, \lambda_2 = 0, D^{m,k}_{\lambda_1,\lambda_2}$ reduces to D^m which is introduced by Salagean [8].

2. when $\lambda_2 = 0, D^{m,k}_{\lambda_1,\lambda_2}$ reduces to D^m_λ which is introduced by Al- Oboudi [1].

Now we will introduce the following definition
Definition 1.1 : Let $UCV(\alpha, \beta, m, k, \lambda_1, \lambda_2)$ be a subclass of S consisting of all functions in S and satisfying the following analytic criterion

$$\Re\left\{ \frac{z(D^{m,k}_{\lambda_1,\lambda_2} f(z))''}{(D^{m,k}_{\lambda_1,\lambda_2} f(z))'} + 1 - \alpha \right\} \geq \beta \left| \frac{z(D^{m,k}_{\lambda_1,\lambda_2} f(z))''}{(D^{m,k}_{\lambda_1,\lambda_2} f(z))'} \right|, \quad z \in \mathcal{U} \tag{3}$$

where $-1 \leq \alpha < 1$ and $\beta \geq 0, \lambda_2 \geq \lambda_1 \geq 0, m$ fixed positive natural number.

Let $UCT(\alpha, \beta, m, k, \lambda_1, \lambda_2)$ be a subclass consisting of all functions of S of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0, \quad \forall k \geq 2 \tag{4}$$

and satisfying (3).

In this paper, we obtain a sufficient condition for a function f given by (2) to be in the subclass $UCV(\alpha, \beta, m, k, \lambda_1, \lambda_2)$ and a necessary and sufficient condition for the function given by (4) to be in the subclass $UCT(\alpha, \beta, m, k, \lambda_1, \lambda_2)$, also we derive distortion bounds, extreme points, radii of starlikeness and convexity and integral mean inequalities.

2 Coefficient Estimates

Theorem 2.1 : A function $f(z)$ defined by (2) is said to be in the class $UCV(\alpha, \beta, m, k, \lambda_1, \lambda_2)$ if

$$\sum_{k=2}^{\infty} k[k(1+\beta) - (\alpha + \beta)](\frac{1 + (\lambda_1 + \lambda_2)(k-1)}{1 + \lambda_2(k-1)})^m |a_k| \leq 1 - \alpha, \tag{5}$$

where $-1 \leq \alpha < 1$ and $\beta \geq 0, \lambda_2 \geq \lambda_1 \geq 0, m$ fixed positive natural number.

Proof:

We want to show that

$$\Re\left\{\frac{z(D^{m,k}_{\lambda_1,\lambda_2}f(z))''}{(D^{m,k}_{\lambda_1,\lambda_2}f(z))'}+1-\alpha\right\} \geq \beta\left|\frac{z(D^{m,k}_{\lambda_1,\lambda_2}f(z))''}{(D^{m,k}_{\lambda_1,\lambda_2}f(z))'}\right|,$$

or

$$\beta\left|\frac{z(D^{m,k}_{\lambda_1,\lambda_2}f(z))''}{(D^{m,k}_{\lambda_1,\lambda_2}f(z))'}\right| - \Re\left\{\frac{z(D^{m,k}_{\lambda_1,\lambda_2}f(z))''}{(D^{m,k}_{\lambda_1,\lambda_2}f(z))'}\right\} \leq 1-\alpha$$

$$\beta\left|\frac{z(z+\sum_{k=2}^{\infty}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^k)''}{(z+\sum_{k=2}^{\infty}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^k)'}\right| - \Re\left\{\frac{z(z+\sum_{k=2}^{\infty}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^k)''}{(z+\sum_{k=2}^{\infty}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^k)'}\right\} \leq 1-\alpha$$

$$\beta\left|\frac{\sum_{k=2}^{\infty}k(k-1)(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^{k-1}}{(1+\sum_{k=2}^{\infty}k(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^{k-1}}\right| - \Re\left\{\frac{\sum_{k=2}^{\infty}k(k-1)(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^{k-1}}{(1+\sum_{k=2}^{\infty}k(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^{k-1}}\right\} \leq 1-\alpha$$

$$\frac{(1+\beta)\sum_{k=2}^{\infty}k(k-1)(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m |a_k|}{1+\sum_{k=2}^{\infty}k(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m |a_k|} \leq 1-\alpha$$

$$\sum_{k=2}^{\infty}k[k(1+\beta)-(\alpha+\beta)](\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m |a_k| \leq 1-\alpha,$$

by (5) we get the result.

Hence the proof is complete.

Theorem 2.2 : A necessary and sufficient condition for a function $f(z)$ of the form (4) to be in the class $UCT(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ is that

$$\sum_{k=2}^{\infty}k[k(1+\beta)-(\alpha+\beta)](\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k \leq 1-\alpha, \tag{6}$$

where $-1 \leq \alpha < 1$ and $\beta \geq 0, \lambda_2 \geq \lambda_1 \geq 0, m$ fixed positive natural number.

Proof:

To prove the necessary part, let $f(z) \in UCT(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ and z be a real. Then

$$\Re\left\{\frac{z(D^{m,k}_{\lambda_1,\lambda_2}f(z))''}{(D^{m,k}_{\lambda_1,\lambda_2}f(z))'}+1-\alpha\right\} \geq \beta\left|\frac{z(D^{m,k}_{\lambda_1,\lambda_2}f(z))''}{(D^{m,k}_{\lambda_1,\lambda_2}f(z))'}\right|, \quad z \in \mathcal{U} \tag{3}$$

which gives

$$\beta\left|\sum_{k=2}^{\infty}k(k-1)(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^{k-1}\right| + \sum_{k=2}^{\infty}k(k-1)(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^{k-1}$$

$$+(1-\alpha)\sum_{k=2}^{\infty}k(k-1)(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k z^{k-1}\leq 1-\alpha.$$

Letting $z\rightarrow^{-}1$ along the real axis, we get

$$\sum_{k=2}^{\infty}k[k(1+\beta)-(\alpha+\beta)](\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k\leq 1-\alpha,$$

which is the required inequality.
The sufficient part is true by Theorem 2.1.
Hence the proof is complete.

Corollary 2.1 : Let the function $f(z)$ defined by (4) be in the subclass $UCT(\alpha,\beta,m,k,\lambda_1,\lambda_2)$. Then

$$a_k\leq\frac{(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m, k\geq 2$$

Remark 2.1 : We can introduce a new subclass of $UCT(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ by fixing the second coefficient of the function $f(z)$ of the form (4) in $UCT(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ by using Theorem 2 as follows,

$$a_2=\frac{1-\alpha}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m.$$

Let $UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ denote the class of all functions $f(z)$ in $UCT(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ and has the form

$$f(z)=z-\frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2-\sum_{k=3}^{\infty}a_k z^k, \qquad (7)$$

($a_k\geq 0, 0\leq b\leq 1, -1\leq\alpha\leq 1$ and $\beta\geq 0, \lambda_2\geq\lambda_1\geq 0, m$ fixed positive natural number).

Theorem 2.3 : A function $f(z)\in UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ if and only if

$$\sum_{k=3}^{\infty}k[k(1+\beta)-(\alpha+\beta)](\frac{(1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k\leq(1-b)(1-\alpha),$$

where $-1\leq\alpha<1$ and $\beta\geq 0, 0\leq b\leq 1, \lambda_2\geq\lambda_1\geq 0, m$ fixed positive natural number.

Proof:
By using Theorem 2.2, we have:

$$\sum_{k=2}^{\infty}k[k(1+\beta)-(\alpha+\beta)](\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k\leq 1-\alpha$$

$$2(2+\beta-\alpha)(\frac{1+\lambda_1+\lambda_2}{1+\lambda_2})^m a_2+\sum_{k=3}^{\infty}k[k(1+\beta)-(\alpha+\beta)](\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k\leq 1-\alpha \qquad (8)$$

$$a_2 = \frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m$$

Substitute it in (8) we get

$$\sum_{k=3}^{\infty} k[k(1+\beta)-(\alpha+\beta)](\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k \leq (1-b)(1-\alpha)$$

Hence the proof is complete.

Corollary 2.2 : Let the function $f(z)$ defined by (4) be in the subclass $UCT_b(\alpha, \beta, m, k, \lambda_1, \lambda_2)$. Then

$$a_k \leq \frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m, \quad k \geq 3$$

$-1 \leq \alpha < 1$ and $\beta \geq 0, 0 \leq b \leq 1, \lambda_2 \geq \lambda_1 \geq 0, m$ fixed positive natural number.

3 Distortion Theorem

Theorem 3.1 : Let the function f defined by (4) be in the class $UCT_b(\alpha, \beta, m, k, \lambda_1, \lambda_2)$. Then

$$|z| - \frac{(1-\alpha)b}{2(2+\beta-\alpha))}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m |z|^2 - \frac{(1-b)(1-\alpha)}{3[3+2\beta-\alpha]}(\frac{1+3\lambda_2}{1+3(\lambda_1+\lambda_2)})^m |z|^3 \leq |f(z)|$$

$$\leq |z| + \frac{(1-\alpha)b}{2(2+\beta-\alpha))}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m |z|^2 + \frac{(1-b)(1-\alpha)}{3[3+2\beta-\alpha]}(\frac{1+3\lambda_2}{1+3(\lambda_1+\lambda_2)})^m |z|^3$$

Proof:
Since $f \in UCT_b(\alpha, \beta, m, k, \lambda_1, \lambda_2)$, then by Corollary 2.2 we have

$$a_k \leq \frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m, k \geq 2$$

For $0 \leq |z| = r < 1$

$$|f(z)| = \left| z - \frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2 - \sum_{k=3}^{\infty} a_k z^k \right|$$

$$\leq |z| + \frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m |z|^2 + \sum_{k=3}^{\infty} a_k |z|^k$$

$$\leq |z| + \frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m |z|^2 + \frac{(1-b)(1-\alpha)}{3[3+2\beta-\alpha]}(\frac{1+3\lambda_2}{1+3(\lambda_1+\lambda_2)})^m |z|^3,$$

and

$$|f(z)| \geq |z| - \frac{(1-\alpha)b}{2(2+\beta-\alpha))}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m |z|^2 - \frac{(1-b)(1-\alpha)}{3[3+2\beta-\alpha]}(\frac{1+3\lambda_2}{1+3(\lambda_1+\lambda_2)})^m |z|^3.$$

This completes the proof.

4 Convex Linear Combination

Theorem 4.1 : The class $UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ is convex set.

Proof :Let

$$f(z)=z-\frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2-\sum_{k=3}^{\infty}a_k z^k,\quad (a_k\geq 0)$$

$$g(z)=z-\frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2-\sum_{k=3}^{\infty}b_k z^k,\quad (b_k\geq 0, 0\leq b\leq 1)$$

be in the class $UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$. Let

$$G(z)=\lambda f(z)+(1-\lambda)g(z),\quad 0\leq\lambda\leq 1.$$

We must prove that $G(z)\in UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$,

$$G(z)=z-\frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2-\sum_{k=3}^{\infty}(a_k\lambda+b_k(1-\lambda))z^k,$$

by using Theorem 2.3, we get:

$$\sum_{k=3}^{\infty}k[k(1+\beta)-(\alpha+\beta)](\frac{(1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m(a_k\lambda+b_k(1-\lambda))$$

$$\leq(\lambda+(1-\lambda))(1-b)(1-\alpha)=(1-b)(1-\alpha).$$

Therefore $G(z)\in UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$.
Thus the class $UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ is convex set.
Hence the proof is complete.

Theorem 4.2 : Let

$$f_i(z)=z-\frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2-\sum_{k=3}^{\infty}a_{k,i}z^k,\quad (a_{k,i}\geq 0)$$

be in the class $UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ for every (i= 1,2, ..., n).
Then the function

$$F(z)=\sum_{i=1}^{n}\lambda_i f_i(z),$$

where

$$\sum_{i=1}^{n}\lambda_i=1,$$

is also in the class $UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$.

Proof : We have

$$F(z)=\sum_{i=1}^{n}\lambda_i f_i(z),$$

where $\sum_{i=1}^{n}\lambda_i = 1$

$$F(z) = z - \frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2 - \sum_{k=3}^{\infty}(\sum_{i=1}^{n} a_{k,i}\lambda_i)z^k,$$

by using Theorem 2.3 we get:

$$\sum_{k=3}^{\infty} k[k(1+\beta)-(\alpha+\beta)](\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m(\sum_{i=1}^{n} a_{k,i}\lambda_i)$$

$$= \sum_{i=1}^{n}\lambda_i \sum_{k=3}^{\infty} k[k(1+\beta)-(\alpha+\beta)](\frac{1+(\lambda_1+\lambda_2)(k-1)}{(1+\lambda_2)(k-1)})^m a_{k,i}$$

$$\leq \sum_{i=1}^{n}\lambda_i(1-b)(1-\alpha) = (1-b)(1-\alpha).$$

Thus $F(z) \in UCT_b(\alpha, \beta, m, k, \lambda_1, \lambda_2)$.

Theorem 4.3 : Let

$$f_2(z) = z - \frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2,$$

and

$$f_k(z) = z - \frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2 - \frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}$$

$$(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m z^k$$

for $k = 3, 4, ...$ Then $f(z) \in UCT_b(\alpha, \beta, m, k, \lambda_1, \lambda_2)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=2}^{\infty}\lambda_k f_k(z), \tag{9}$$

where $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty}\lambda_k = 1$.

Proof : Suppose that $f(z)$ can be expressed in the form (9). Then

$$f(z) = z - \frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2$$

$$-\sum_{k=3}^{\infty}\lambda_k\frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m z^k,$$

put

$$a_k = \frac{\lambda_k(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m,$$

by Theorem 2.3, we have:

$$\sum_{k=3}^{\infty}\frac{k[k(1+\beta)-(\alpha+\beta)]}{(1-b)(1-\alpha)}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a^k$$

$$=\sum_{k=3}^{\infty}\lambda_k=1-\lambda_2<1,$$

Thus $f\in UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$.
Conversely, suppose that $f(z)\in UCT_b(\alpha,\beta,m,k,\lambda_1,\quad\lambda_2)$

$$f(z)=z-\frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2-\sum_{k=3}^{\infty}a_k z^k.$$

Let

$$\lambda_k=\frac{k[k(1+\beta)-(\alpha+\beta)]}{(1-b)(1-\alpha)}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m a_k,$$

and

$$1-\sum_{k=3}^{\infty}\lambda_k=\lambda_2.$$

Then

$$f(z)=\sum_{k=2}^{\infty}\lambda_k f_k(z). \tag{9}$$

This completes the proof of the theorem.

Corollary 4.1 : The extreme points of the class $UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ are the functions

$$f_2(z)=z-\frac{b(1-\alpha)}{2(2+\beta-\alpha)+\beta)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2,$$

and

$$f_k(z)=z-\frac{b(1-\alpha)}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z^2-\frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]},$$

$$(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m z^k,$$

for $k=3,4,....$
Now we find the radii of starlikeness, convexity and close-to-convexity for the functions in the class $UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$.

5 Radii of Starlikness, Convexity and Close-to-Convexity

Theorem 5.1 : Let $f\in UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$. Then f is starlike of order $\delta(0\leq\delta<1)$ in $|z|<r$, where r is the largest value such that

$$\frac{(2-\delta)(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m r+\sum_{k=3}^{\infty}\frac{(k-\delta)(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m r^{k-1}<1-\delta.$$

Proof:
If f is starlike of order δ, then

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq 1-\delta$$

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{-\frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z-\sum_{k=3}^{\infty}(k-1)a_k z^{k-1}}{1-\frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z-\sum_{k=3}^{\infty}a_k z^{k-1}}\right|\leq 1-\delta.$$

Thus

$$\frac{(2-\delta)(1-\alpha)b}{2(2+\beta-\alpha)+\beta)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m|z|+\sum_{k=3}^{\infty}(k-\delta)a_k|z|^{k-1}<1-\delta.$$

Since

$$a_k\leq\frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m,\quad k\geq 2.$$

Hence

$$\frac{(2-\delta)(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m r+\sum_{k=3}^{\infty}\frac{(k-\delta)(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m r^{k-1}<1-\delta.$$

This completes the proof of the theorem.

Theorem 5.2 : Let $f\in UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$. Then f is convex of order $\delta(0\leq\delta<1)$ in $|z|<r$, where r is the largest value such that

$$\frac{(3-2\delta)(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m r+\sum_{k=3}^{\infty}\frac{(k-\delta)(1-b)(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m r^{k-1}<1-\delta.$$

Proof:
If $f\in UCT_b(\alpha,\beta,m,k,\lambda_1,\lambda_2)$ is convex of order δ, then

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq 1-\delta,\quad 0\leq\delta<1 \tag{10}$$

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{\frac{-(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z-\sum_{k=3}^{\infty}k(k-1)a_k z^{k-1}}{1-\frac{(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z-\sum_{k=3}^{\infty}ka_k z^{k-1}}\right|,$$

inequality (10) is true if the above quantity is less than $1-\delta, 0\leq\delta<1$.
Then

$$\frac{(3-2\delta)(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m r+\sum_{k=3}^{\infty}k(k-\delta)a_k r^{k-1}<1-\delta$$

$$\frac{(3-2\delta)(1-\alpha)b}{2(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m r+\sum_{k=3}^{\infty}\frac{(k-\delta)(1-b)(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m r^{k-1}<1-\delta.$$

Hence the proof is complete.

Theorem 5.3 : Let $f(z) \in UCT_b(\alpha, \beta, m, k, \lambda_1, \lambda_2)$. Then $f(z)$ is close-to-convex of order $\delta (0 \le \delta < 1)$ in $|z| < r$, where r is the largest value such that

$$\frac{(1-\alpha)b}{(2+\beta-\alpha))}\left(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2}\right)^m r + \sum_{k=3}^{\infty} \frac{(1-b)(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]}\left(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)}\right)^m r^{k-1} < 1-\delta.$$

Proof:
If $f \in UCT_b(\alpha, \beta, m, k, \lambda_1, \lambda_2)$ is close-to-convex of order $\delta(0 \le \delta < 1)$, then

$$\left|f'(z) - 1\right| \le 1 - \delta.$$

Now

$$\left|f'(z) - 1\right| = \left|1 - \frac{(1-\alpha)b}{(2+\beta-\alpha)}\left(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2}\right)^m z - \sum_{k=3}^{\infty} k a_k z^{k-1} - 1\right|$$

$$< \frac{(1-\alpha)b}{(2+\beta-\alpha)}\left(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2}\right)^m r + \sum_{k=3}^{\infty} k a_k r^{k-1} < 1 - \delta,$$

by Corollary 2.2, we have

$$a_k \le \frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}\left(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)}\right)^m, k \ge 2.$$

Thus

$$\frac{(1-\alpha)b}{(2+\beta-\alpha))}\left(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2}\right)^m r + \sum_{k=3}^{\infty} \frac{(1-b)(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]}\left(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)}\right)^m r^{k-1} < 1-\delta.$$

Hence the proof is complete.

6 Integral Means Inequalities

In 1925, Littlewood [5] proved the following subordination theorem.

Theorem 6.1 : If f and g are any two functions, analytic in $\mathcal{U}$, with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}, (0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\mu d\theta \le \int_0^{2\pi} |g(z)|^\mu d\theta.$$

Theorem 6.2 : Let the function f defined by (4) be in the class $UCT_b(\alpha, \beta, m, k, \lambda_1, \lambda_2)$ and f_k be defined by

$$f_k(z) = z - \frac{(1-\alpha)b}{2(2+\beta-\alpha))}\left(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2}\right)^m z^2 - \frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}\left(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)}\right)^m z^k,$$

for $k = 3, 4, ...$
If there exists an analytic function $w(z)$ given by

$$[w(z)]^{k-1} = \frac{k[k(1+\beta)-(\alpha+\beta)]}{(1-b)(1-\alpha)}\left(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)}\right)^m \sum_{k=3}^{\infty} a_k z^{k-1},$$

then for $z = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |f(re^{i\theta})|^\mu d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\mu d\theta, \quad (\mu > 0).$$

Proof:
We must prove that

$$\int_0^{2\pi} |1 - \frac{(1-\alpha)b}{(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z - \sum_{k=3}^{\infty} a_k z^{k-1}|^\mu d\theta \leq$$

$$\int_0^{2\pi} |1 - \frac{(1-\alpha)b}{(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z - \frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m z^{k-1}|^\mu d\theta|,$$

by Theorem 6.1, it suffices to show that

$$1 - \frac{(1-\alpha)b}{(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z - \sum_{k=3}^{\infty} a_k z^{k-1} \prec 1 - \frac{(1-\alpha)b}{(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z$$

$$- \frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m z^{k-1}.$$

Taking

$$1 - \frac{(1-\alpha)b}{(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z - \sum_{k=3}^{\infty} a_k z^{k-1} =$$

$$1 - \frac{(1-\alpha)b}{(2+\beta-\alpha)}(\frac{1+\lambda_2}{1+\lambda_1+\lambda_2})^m z - \frac{(1-b)(1-\alpha)}{k[k(1+\beta)-(\alpha+\beta)]}(\frac{1+\lambda_2(k-1)}{1+(\lambda_1+\lambda_2)(k-1)})^m [w(z)]^{k-1}.$$

Thus

$$[w(z)]^{k-1} = \frac{k[k(1+\beta)-(\alpha+\beta)]}{(1-b)(1-\alpha)}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m \sum_{k=3}^{\infty} a_k z^{k-1}$$

$w(0) = 0$, and by Theorem 2.3, we have

$$|[w(z)]|^{k-1} = |\frac{k[k(1+\beta)-(\alpha+\beta)]}{(1-b)(1-\alpha)}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m \sum_{k=3}^{\infty} a_k z^{k-1}|$$

$$\leq \frac{k[k(1+\beta)-(\alpha+\beta)]}{(1-b)(1-\alpha)}(\frac{1+(\lambda_1+\lambda_2)(k-1)}{1+\lambda_2(k-1)})^m \sum_{k=3}^{\infty} |a_k||z|^{k-1}$$

$$\leq |z| < 1.$$

Hence the proof is complete.

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